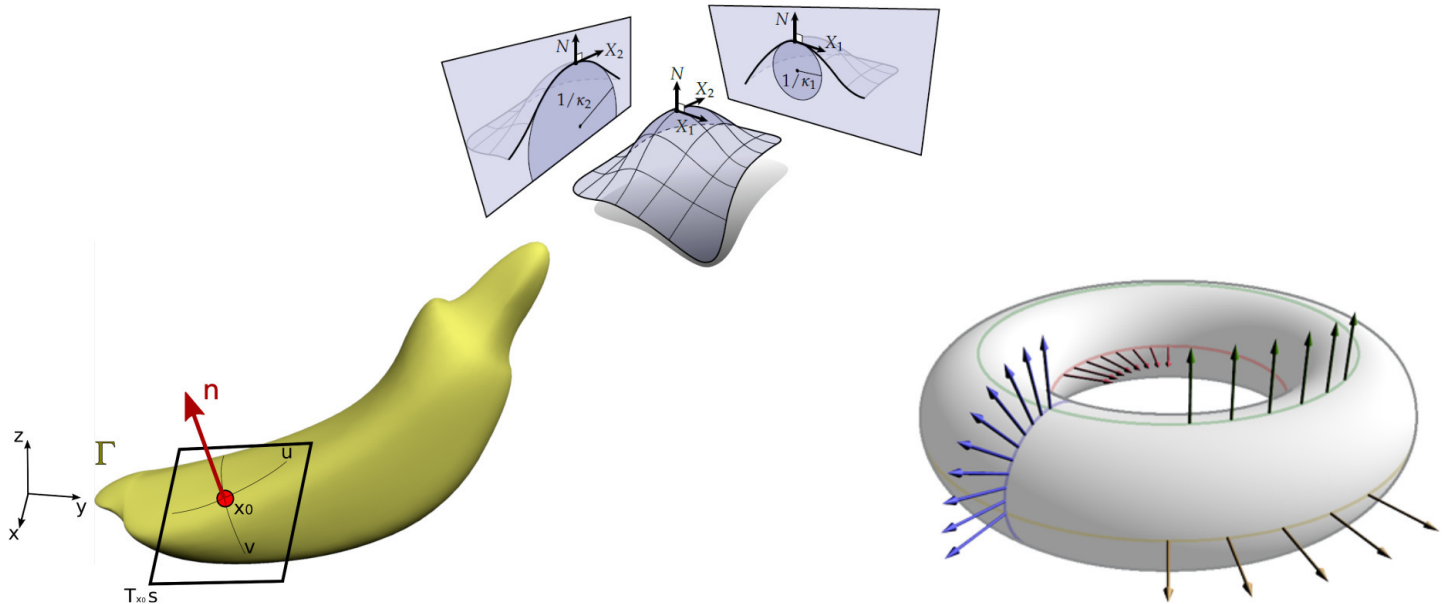


Smooth Surfaces and Differential Geometry

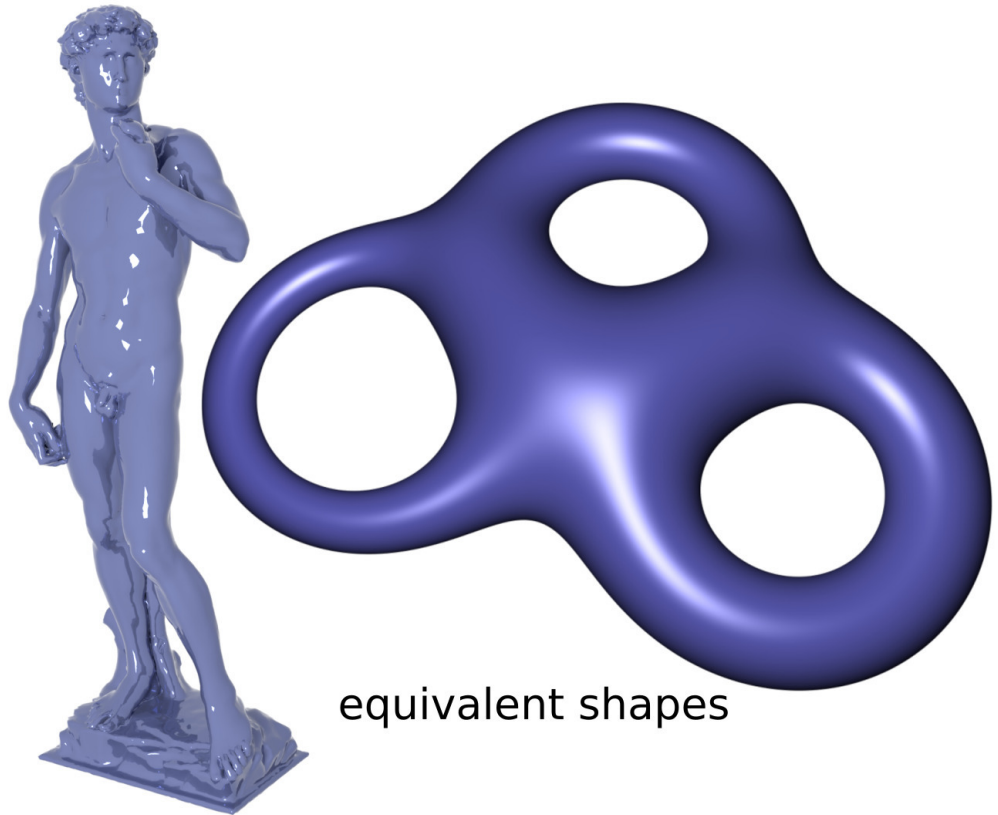


Topology

Topology:

Study of an object independently from its geometry

2 shapes are
topologically equivalent
if you can deform one
onto the other using
only continuous
deformation
(no cutting, no merging)



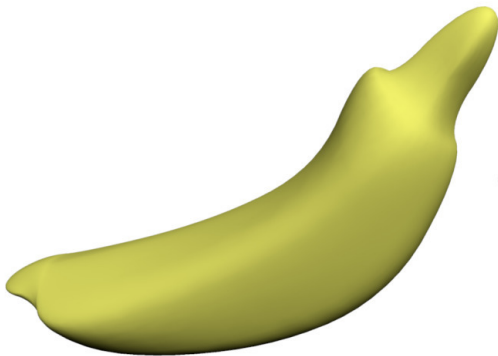
equivalent shapes

Manifold

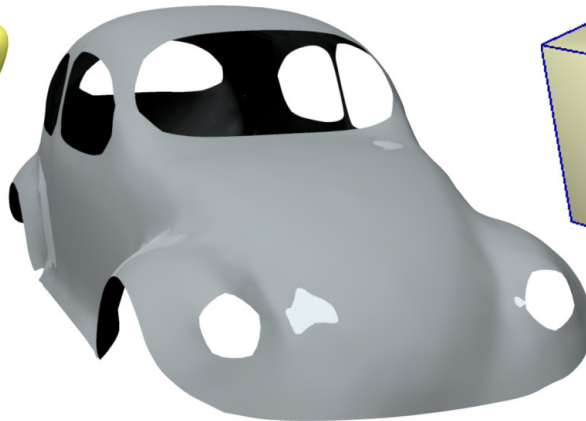
A surface Γ is a **2-manifold** if every point has a neighbor **homeomorphic to a (half) disc**.

Homeomorphism = bijective continuous application with continuous reciprocal function.

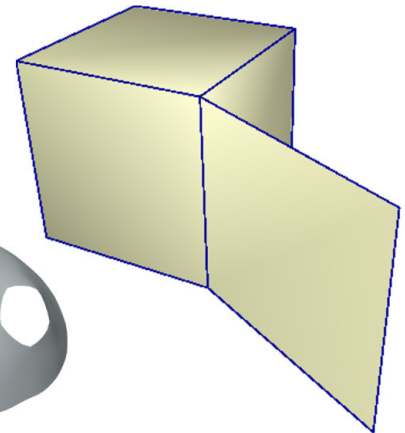
2-manifold



*2-manifold
with boundaries*



not a 2-manifold



Surface continuity

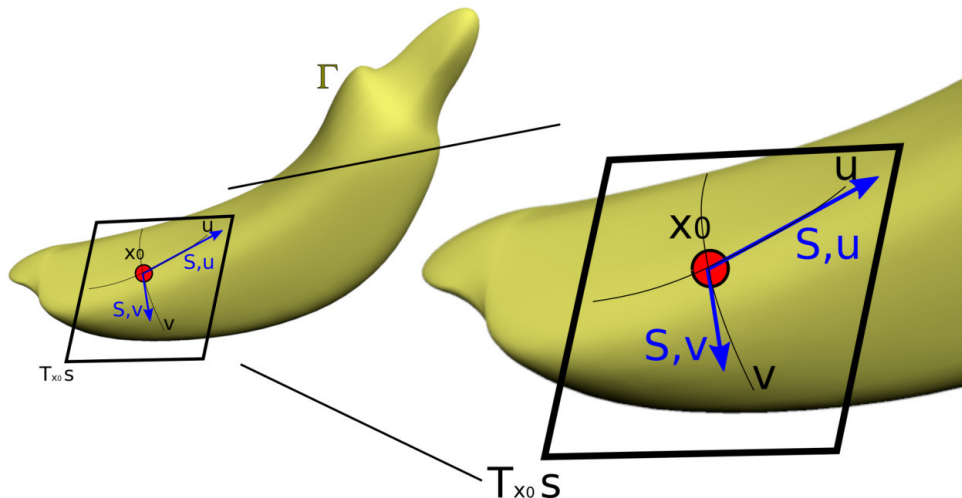
Let Γ be the surface associated to the mapping S

$$S: \begin{cases} \mathcal{D} \subset \mathbb{R}^2 & \rightarrow \mathbb{R}^3 \\ (u, v) & \mapsto S(u, v) \end{cases}$$

$$\Gamma = tr_{\mathcal{D}}(S) = \{\mathbf{x} \in \mathbb{R}^3 \mid \forall (u, v) \in \mathcal{D}, S(u, v) = \mathbf{x}\}$$

S is C^1 if $S_{,u}$ and $S_{,v}$ are defined and continuous.

S is C^2 if $S_{,uu}$, $S_{,vv}$, and $S_{,uv}$ are defined and continuous.



Geometrical continuity

Γ is G^1 if there is a tangent plane in every position.

Γ is G^2 if the curvature of the surface is continuous everywhere.

Note: $S \subset \mathcal{C}^k \neq \Gamma \subset G^k$.

G^2 is necessary for the smooth reflexion.

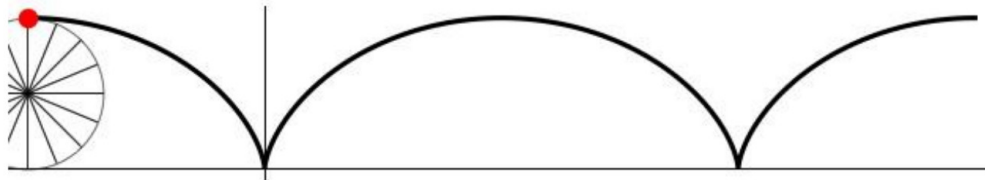
Exemple of continuity differences

$$f: \begin{cases} x(t) = t \\ y(t) = t \end{cases}, t \in [-1, 0[$$

$$\begin{cases} x(t) = t/2 \\ y(t) = t/2 \end{cases}, t \in [0, 2]$$

Are theses functions C^1 , C^2 , G^1 , G^2 ?

$$h: \begin{cases} x(t) = R(t - \sin(t)) \\ y(t) = R(t - \cos(t)) \end{cases}$$



Reminder about curves

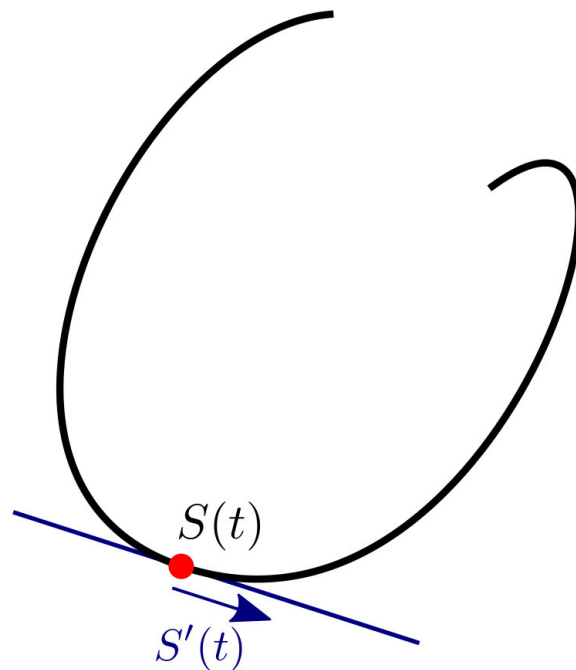
Curve $S(t)$

$$\mathbb{R} \mapsto \mathbb{R}^3$$

$$S : t \rightarrow (S_x(t), S_y(t), S_z(t))$$

Length of a curve:

$$L = \int_t \|S'(t)\| dt$$



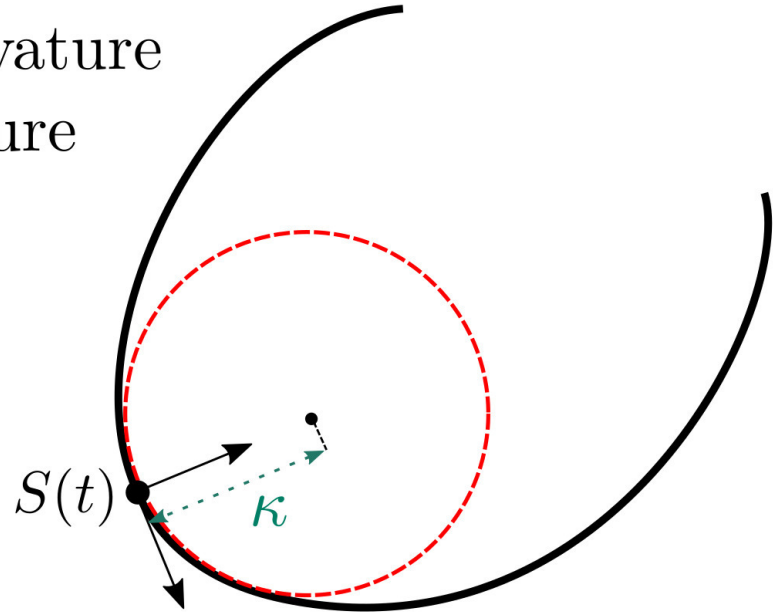
Curvature of a curve

κ is the radius of curvature

$\lambda = 1/\kappa$ is the curvature

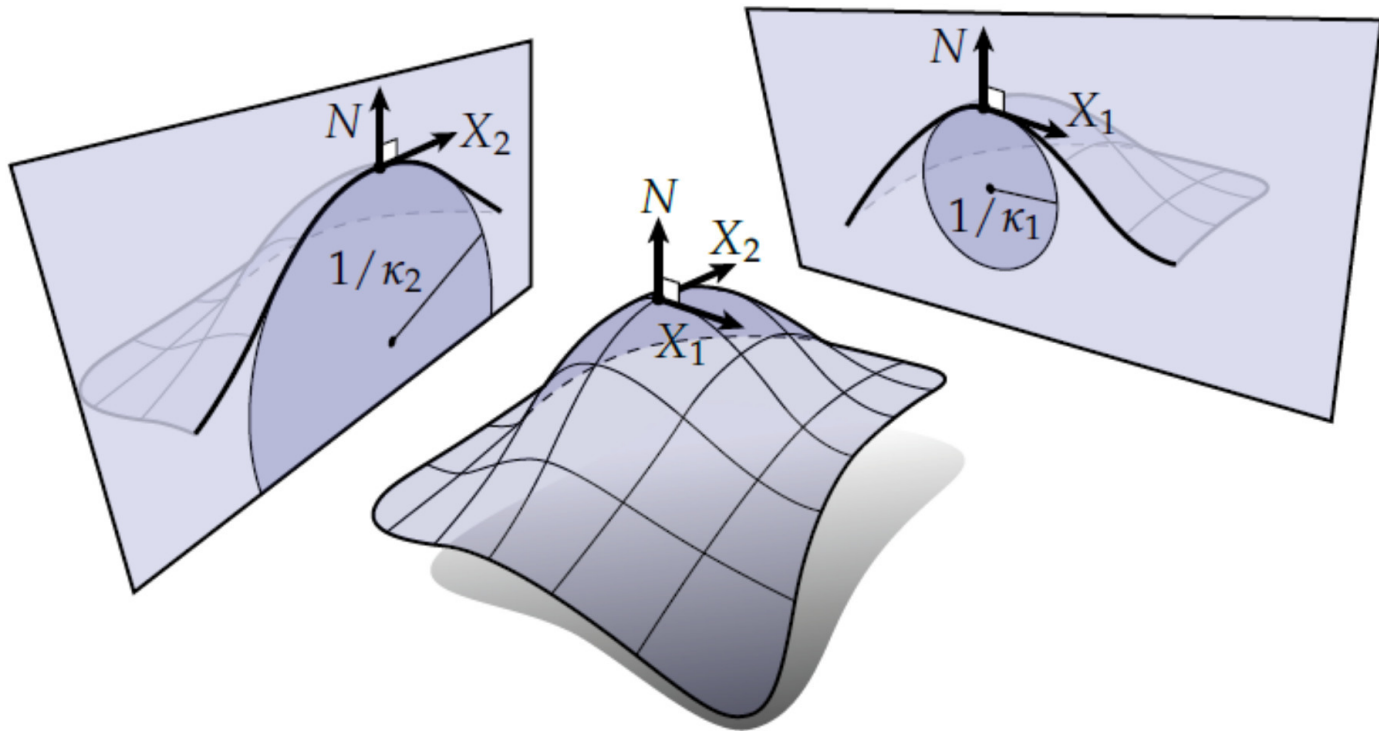
$$\left| \begin{array}{l} \kappa = S'(\rho) \\ \rho = \int \|S'(t)\| \end{array} \right.$$

curvilinear parameterization



$$\kappa = \frac{1}{\|S'(t)\|^3} \left(\|S'(t)\|^2 \|S''(t)\|^2 - \langle S'(t), S''(t) \rangle^2 \right)^{1/2}$$

Intuition for the curvature on a Surface



[Keenan Crane, *Digital Geometry Processing with Discrete Exterior Calculus*, SIGGRAPH 2013]

Tangent plane

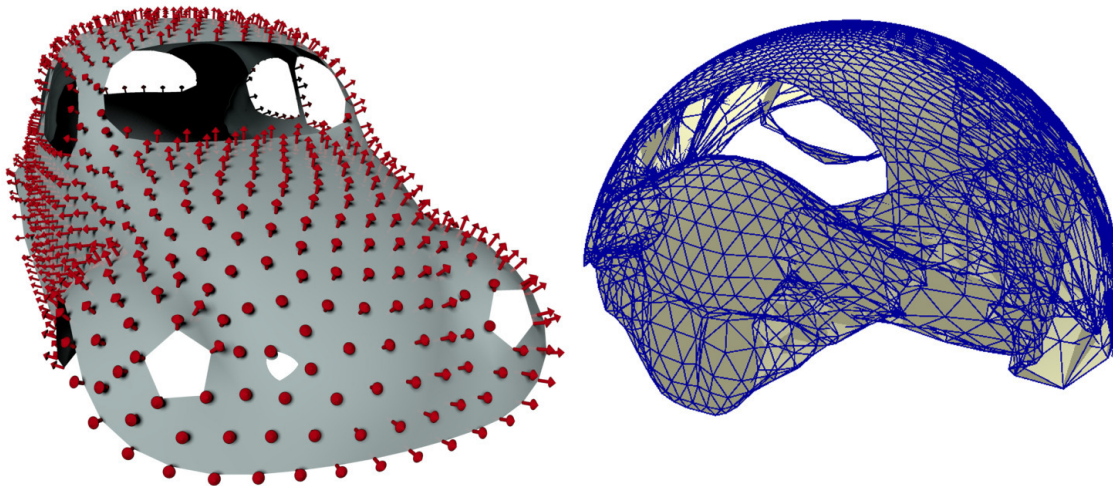
Surface in $\mathbb{R}^3$: $S(u, v) = (S_x(u, v), S_y(u, v), S_z(u, v))$

Normal: $n(u, v) = (S_{,u} \times S_{,v}) / \|S_{,u} \times S_{,v}\| \in \mathbb{S}^2$

Tangent space of S in $x_0 = S(u, v)$

$$T_{x_0}S = \text{Im}(\text{D } S(u, v)) = \{S_{,u}(u, v) h_u + S_{,v}(u, v) h_v \mid (h_u, h_v) \in \mathbb{R}^2\}$$

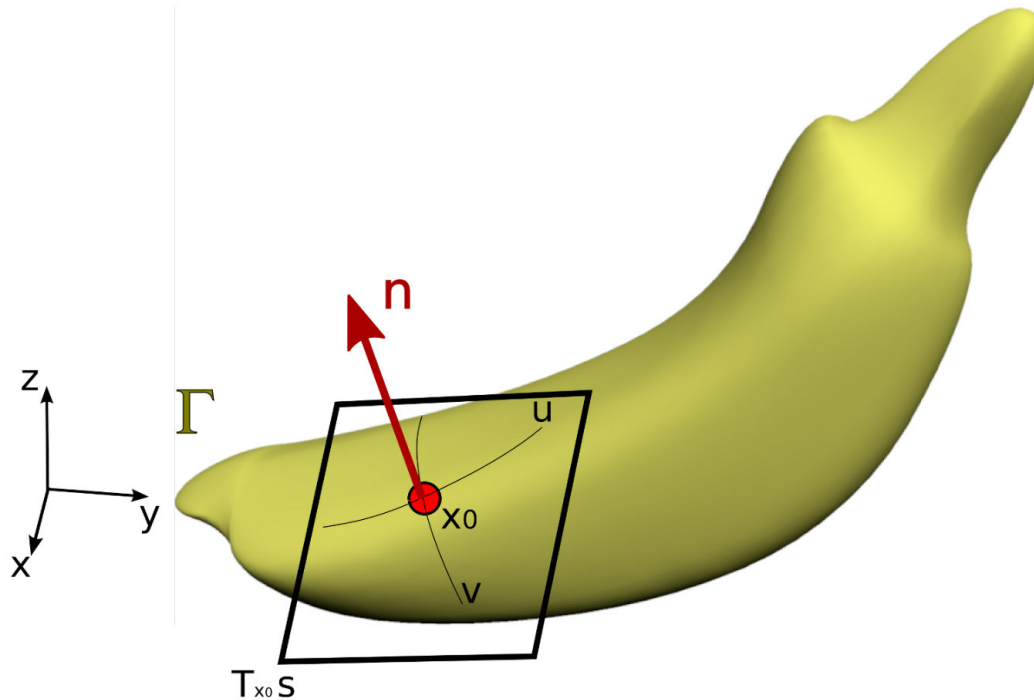
$$N: \begin{cases} \Gamma & \rightarrow \mathbb{S}^2 \\ x_0 = S(u, v) & \mapsto N(x_0) = n(u, v) \end{cases}$$



Integral properties

Area of Γ :
$$\iint_{(u,v) \in \mathcal{D}} \|S_u \times S_v\| \, du \, dv$$

Volume defined by Γ
$$\iint_{(u,v) \in \mathcal{D}} S_z(u, v) n_z^S(u, v) \, du \, dv$$



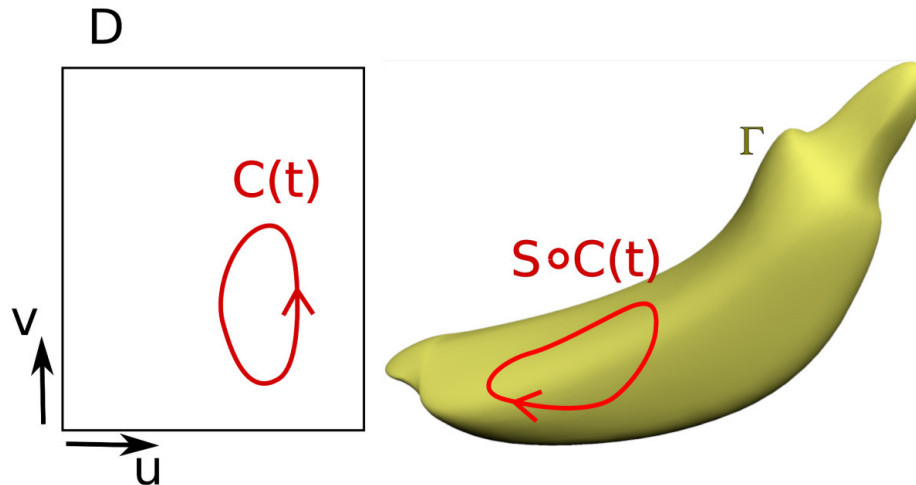
First fundamental form

Curve $C \subset \mathcal{D}$ of length $L = \int_t \langle C'(t), C'(t) \rangle^{1/2} dt$

Length of curve $C_s = S(C_x, C_y)$

$$L_s = \int_t \langle (S \circ C)'(t), (S \circ C)'(t) \rangle^{1/2} dt$$

$$= \int_t (C'^T(t) I_S(t) C'(t))^{1/2} dt$$



First fundamental form

Derivation $L_s = \int_t ((S \circ C)'^T(t) (S \circ C)'(t))^{1/2} dt$

$$(S \circ C)' = C'_x (S_{,u} \circ C) + C'_y (S_{,v} \circ C)$$

$$\begin{aligned} (S \circ C)' &= (S_{,u} \ S_{,v}) \begin{pmatrix} C'_x \\ C'_y \end{pmatrix} \\ &= \partial S^T C' \end{aligned}$$

$$\begin{aligned} (S \circ C)'^T (S \circ C)' &= (C'^T \partial S) (\partial S^T C') \\ &= C'^T (\partial S \partial S^T) C' \\ &= \boxed{C'^T I_s C'} \end{aligned}$$

First fundamental form

I_S First fundamental form / metric tensor

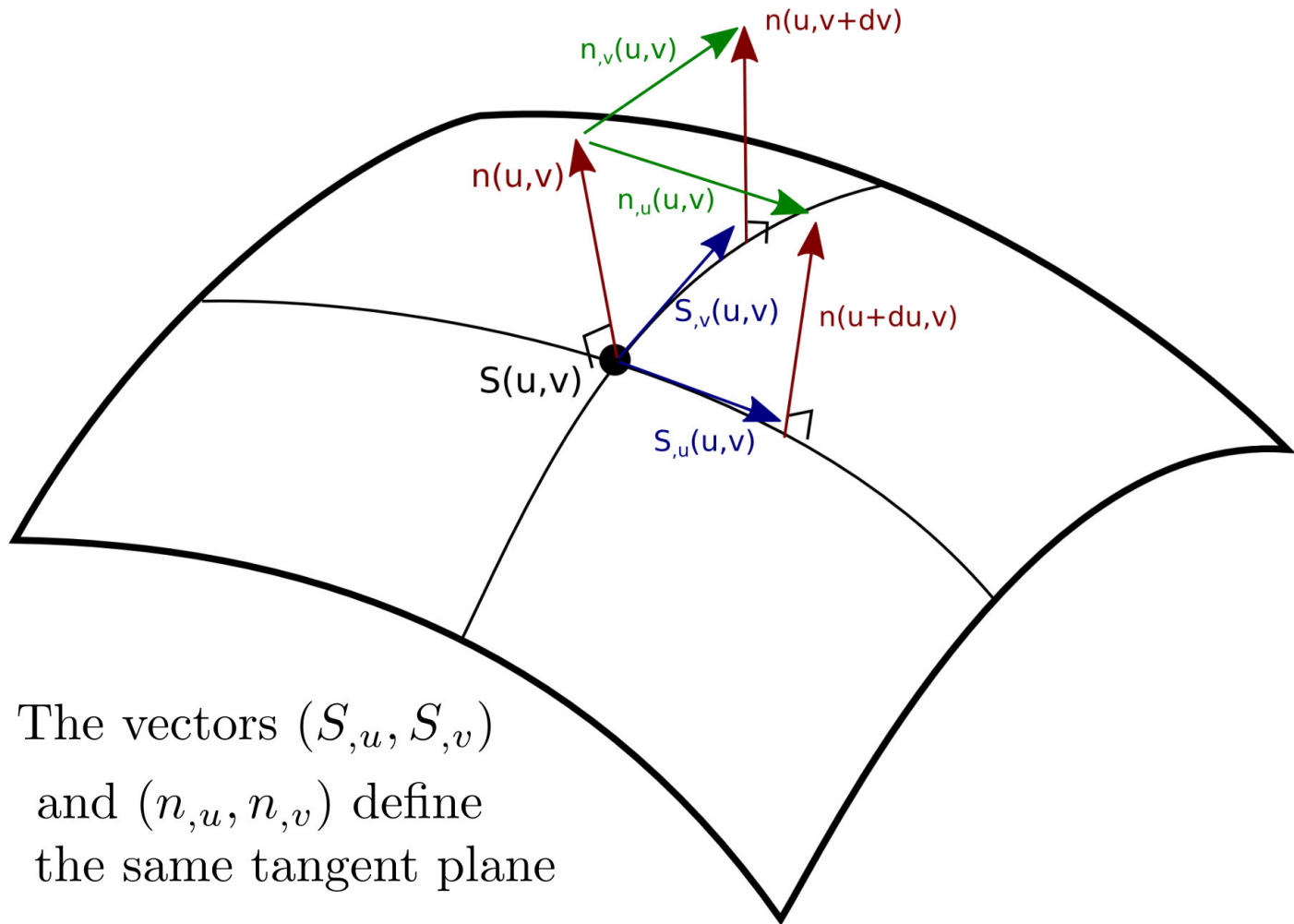
$$I_S = \begin{pmatrix} S_{,u}^2 & \langle S_{,u}, S_{,v} \rangle \\ \langle S_{,u}, S_{,v} \rangle & S_{,v}^2 \end{pmatrix}$$

I_S quadratic form associated to $\langle dS, dS \rangle$

$\sqrt{\det(I_S)}$ = local area change

$$\text{Area of } \Gamma = \iint_{(u,v) \in \mathcal{D}} \sqrt{\det(I_S)} \, du \, dv$$

Derivative of the normals



The vectors (S_u, S_v)
and (n_u, n_v) define
the same tangent plane

Second fundamental form

The vectors $(n_{,u}, n_{,v})$ can be expressed in the $(S_{,u}, S_{,v})$ plane

$$\begin{cases} n_{,u} = w_{00} S_{,u} + w_{01} S_{,v} \\ n_{,v} = w_{10} S_{,u} + w_{11} S_{,v} \end{cases} \quad \begin{pmatrix} n_{,u}^T \\ n_{,v}^T \end{pmatrix} = W_S \begin{pmatrix} S_{,u}^T \\ S_{,v}^T \end{pmatrix}$$

In multiplying both sides by $(S_{,u}, S_{,v})$

$$\mathbb{I}_S = W_S \mathbb{I}_S$$

Where $\mathbb{I}_S$ is the second fundamental form associated to S

$$\mathbb{I}_S = \begin{pmatrix} \langle n_{,u}, S_{,u} \rangle & \langle n_{,u}, S_{,v} \rangle \\ \langle n_{,v}, S_{,u} \rangle & \langle n_{,v}, S_{,v} \rangle \end{pmatrix}$$

Relation with second derivatives

Using orthogonality

$$\begin{cases} \langle n, S_{,u} \rangle = 0 \\ \langle n, S_{,v} \rangle = 0 \end{cases}$$

Differentiating

$$\begin{cases} \langle n_{,u}, S_{,u} \rangle = - \langle n, S_{,uu} \rangle \\ \langle n_{,v}, S_{,u} \rangle = - \langle n, S_{,uv} \rangle \\ \langle n_{,v}, S_{,v} \rangle = - \langle n, S_{,vv} \rangle \end{cases}$$

II can be expressed from n and the second derivatives of S

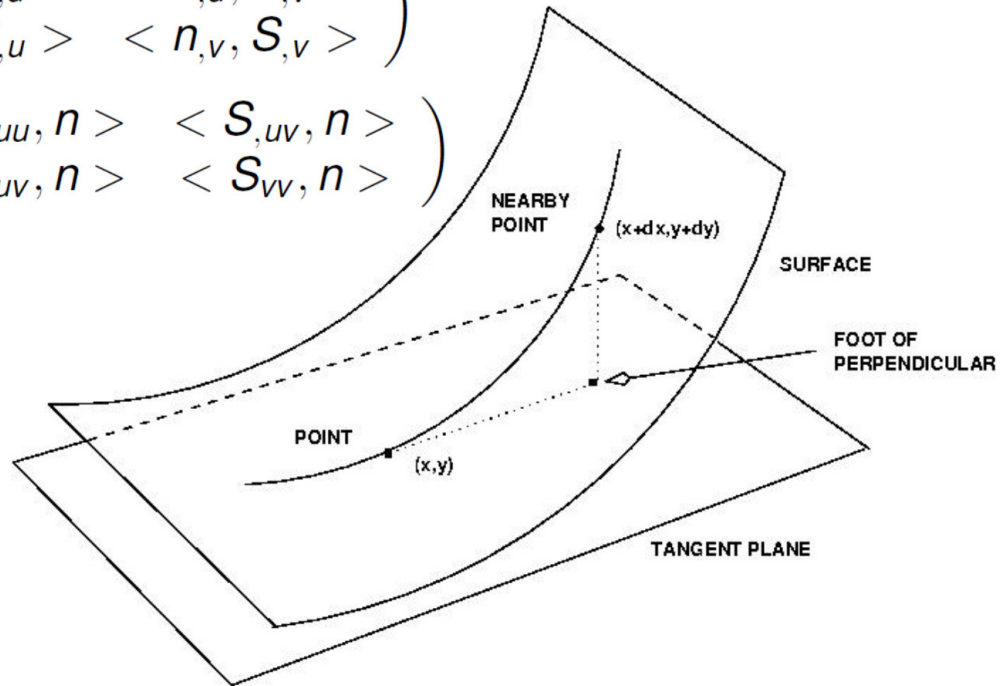
$$\Pi_S = - \begin{pmatrix} \langle n, S_{,uu} \rangle & \langle n, S_{,uv} \rangle \\ \langle n, S_{,uv} \rangle & \langle n, S_{,vv} \rangle \end{pmatrix}$$

Second fundamental form

II is the quadratic form associated to $\langle sS, dn \rangle$
= Taylor expansion of S in its tangent plane

$$\text{II}_S = \begin{pmatrix} \langle n_u, S_u \rangle & \langle n_u, S_v \rangle \\ \langle n_v, S_u \rangle & \langle n_v, S_v \rangle \end{pmatrix}$$

$$\Leftrightarrow \text{II}_S = - \begin{pmatrix} \langle S_{uu}, n \rangle & \langle S_{uv}, n \rangle \\ \langle S_{uv}, n \rangle & \langle S_{vv}, n \rangle \end{pmatrix}$$



[Wikipedia]

Weingarten application

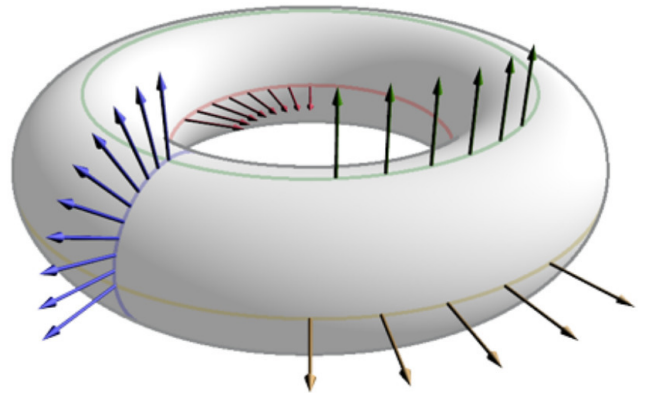
The matrix W_S such that $W_S = \Pi_S I_S^{-1}$
is the Weingarten matrix (or Shape operator)

W_S is diagonalizable and has real eigenvalues

$$W_S = V^T \Lambda V \quad \text{with} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

λ_1, λ_2 are the principal curvature

Weingarten map =
differential of the Gauss map



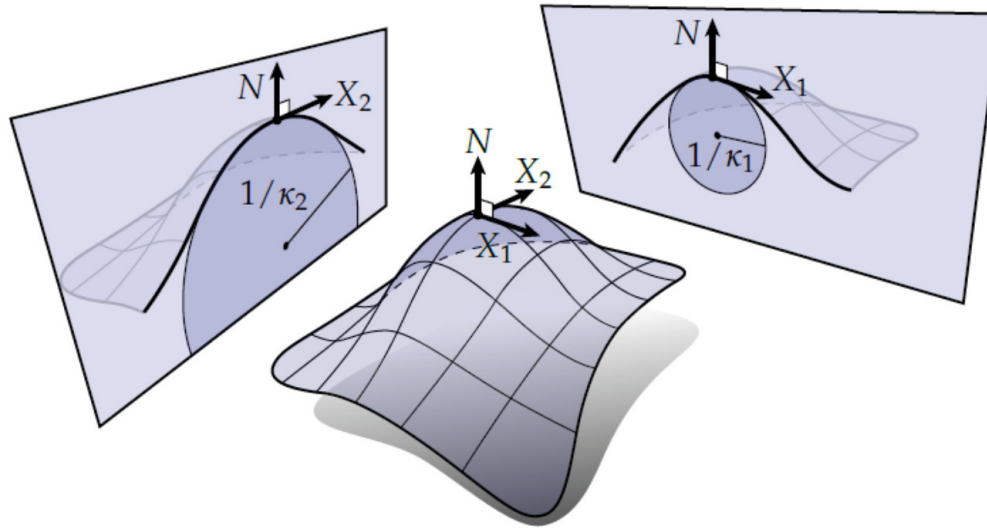
Principal curvatures

Eigenvalues of W_S : principal curvatures (λ_1, λ_2) .

Principal radius of curvatures $(\kappa_1 = \lambda_1^{-1}, \kappa_2 = \lambda_2^{-1})$.

Eigenvectors of W_S :

direction $(\mathbf{v}_1, \mathbf{v}_2)$ of the principal curvatures.



[Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus, SIGGRAPH 2013]

Curvature types

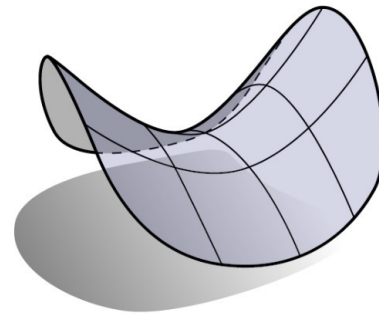
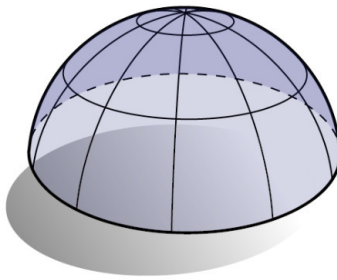
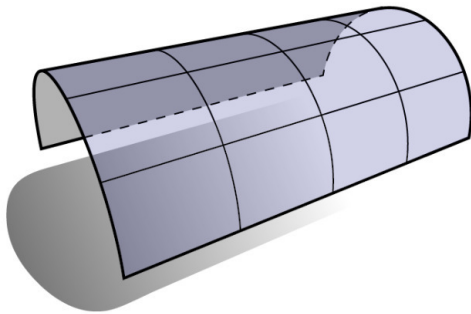
A surface can be locally

Planar: $\lambda_1 = \lambda_2 = 0$

Cylindrical: $\lambda_i \neq 0, \lambda_j = 0$

Elliptic: $\lambda_i \lambda_j > 0$

Hyperbolic: $\lambda_i \lambda_j < 0$



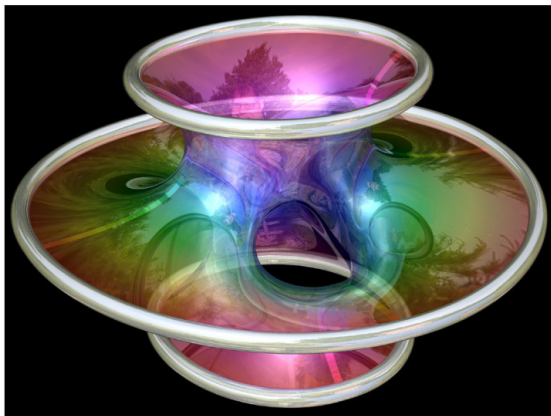
Gauss and mean curvatures

Gauss curvature: $K_S = \lambda_1 \lambda_2 = \det(W_S) = \frac{\det(II_S)}{\det(I_S)}$

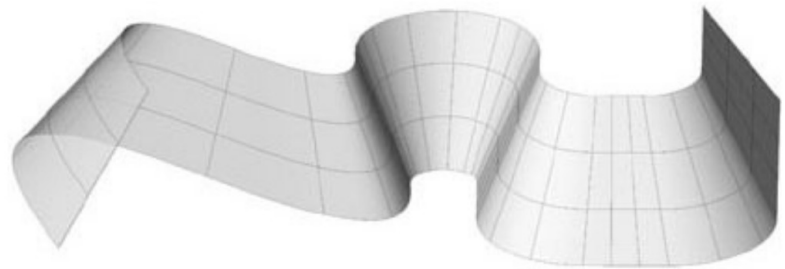
Mean curvature: $H_S = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\text{tr}(W_S)$

$H_S = 0 \Leftrightarrow S$ is a minimal surface

$K_S = 0 \Leftrightarrow S$ is a developable surface



[Paul Nylander]



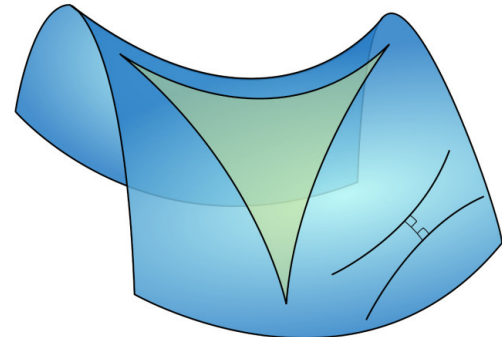
[M. Nettelbladt]

Integral relation

Gauss-Bonnet theorem: $\int_S K_s \, dA + \int_{\partial S} k_g \, ds = 2\pi\chi(S)$

k_g : geodesic curvature

χ : Euler characteristic
(topological invariant)



[Wikipedia]

Application to a mesh around a vertex

$$K \simeq \frac{1}{A} \left(\sum_i \theta_i - 2\pi \right) \Gamma$$

