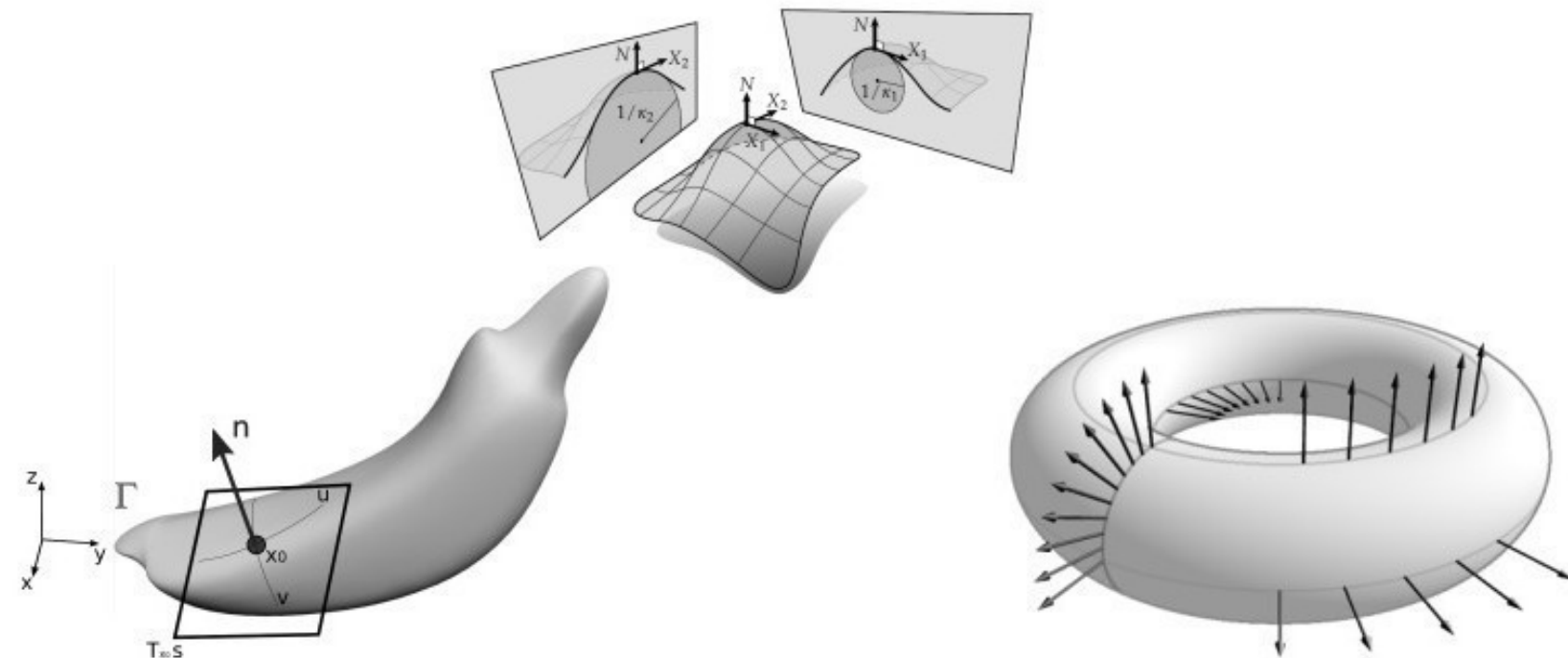


Smooth Surfaces and Differential Geometry



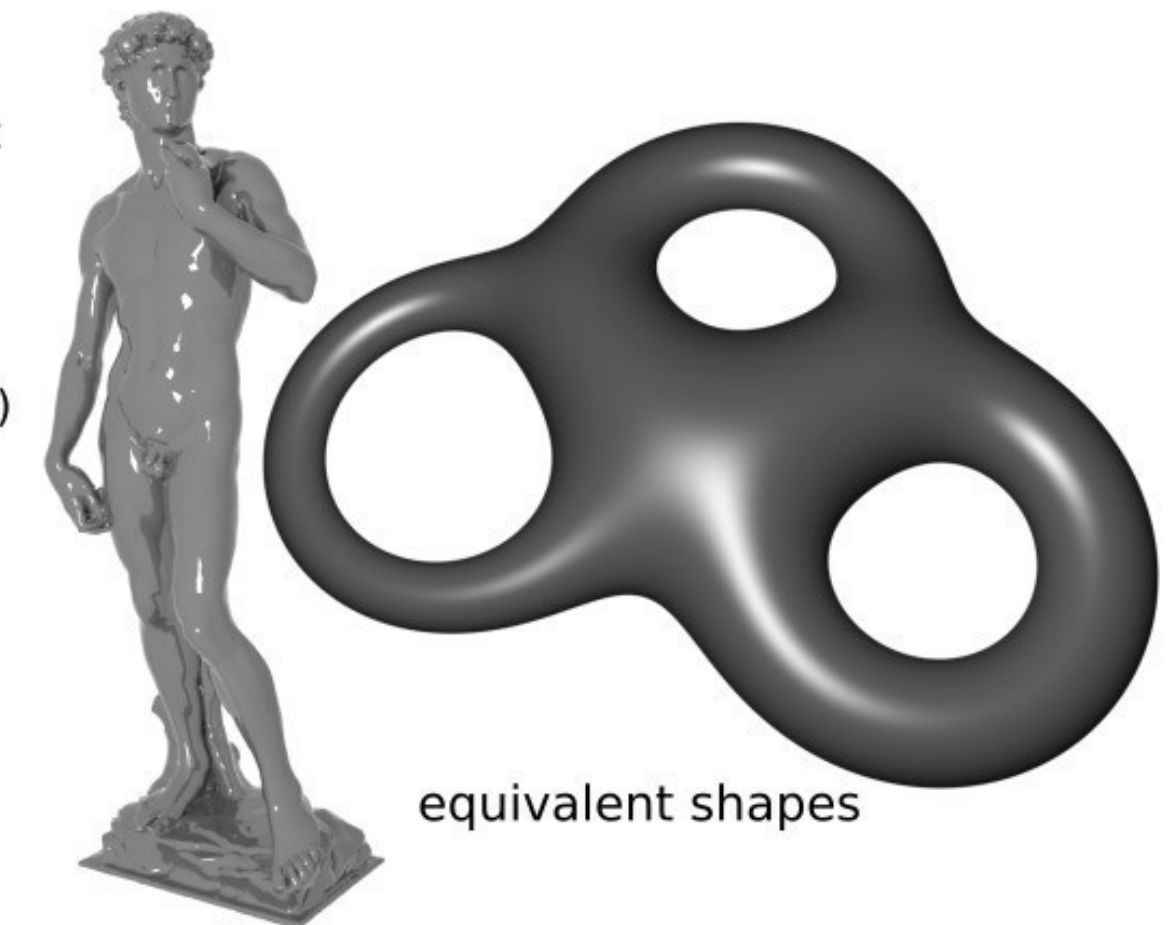
000

Topology

Topology:

Study of an object independently from its geometry

2 shapes are topologically equivalent if you can deform one onto the other using only continuous deformation (no cutting, no merging)

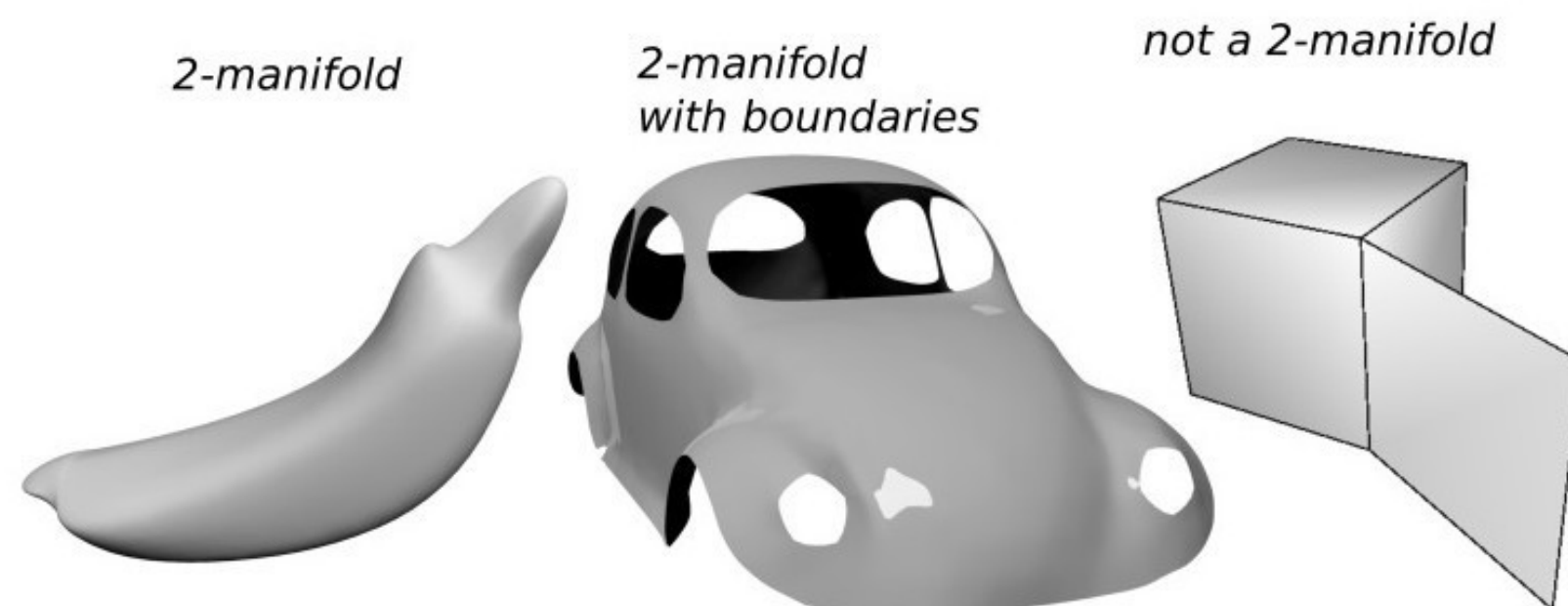


001

Manifold

A surface Γ is a **2-manifold** if every point has a neighbor **homeomorphic to a (half) disc**.

Homeomorphism = bijective continuous application with continuous reciprocal function.



002

Surface continuity

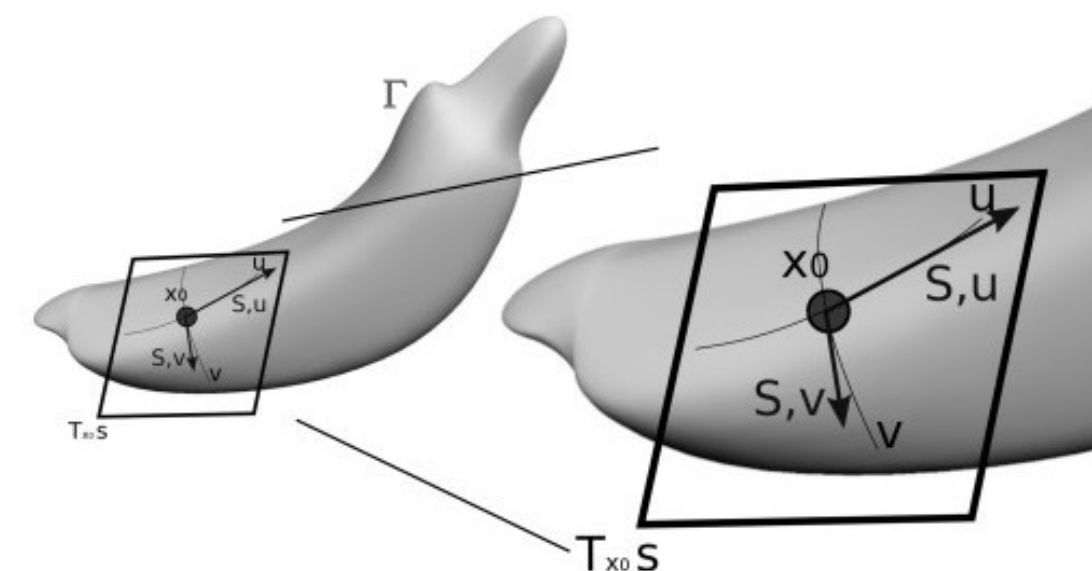
Let Γ be the surface associated to the mapping S

$$S: \begin{cases} \mathcal{D} \subset \mathbb{R}^2 & \rightarrow \mathbb{R}^3 \\ (u, v) & \mapsto S(u, v) \end{cases}$$

$$\Gamma = tr_{\mathcal{D}}(S) = \{\mathbf{x} \in \mathbb{R}^3 | \forall (u, v) \in \mathcal{D}, S(u, v) = \mathbf{x}\}$$

S is C^1 if $S_{,u}$ and $S_{,v}$ are defined and continuous.

S is C^2 if $S_{,uu}$, $S_{,vv}$, and $S_{,uv}$ are defined and continuous.



003

Geometrical continuity

Γ is G^1 if there is a tangent plane in every position.

Γ is G^2 if the curvature of the surface is continuous everywhere.

Note: $S \subset C^k \neq \Gamma \subset G^k$.

G^2 is necessary for the smooth reflexion.

004

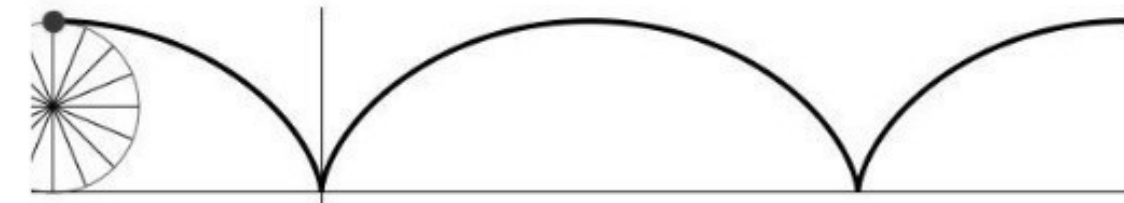
Exemple of continuity differences

$$f: \begin{cases} x(t) = t \\ y(t) = t \end{cases}, t \in [-1, 0[$$

$$\begin{cases} x(t) = t/2 \\ y(t) = t/2 \end{cases}, t \in [0, 2]$$

Are theses functions C^1 , C^2 , G^1 , G^2 ?

$$h: \begin{cases} x(t) = R(t - \sin(t)) \\ y(t) = R(t - \cos(t)) \end{cases}$$



005

Reminder about curves

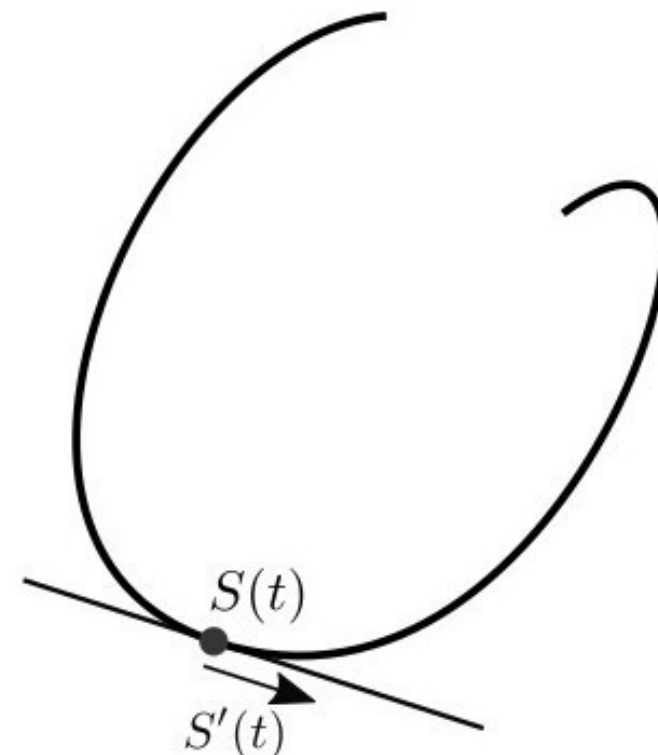
Curve $S(t)$

$$\mathbb{R} \mapsto \mathbb{R}^3$$

$$S: t \rightarrow (S_x(t), S_y(t), S_z(t))$$

Length of a curve:

$$L = \int_t \|S'(t)\| dt$$



006

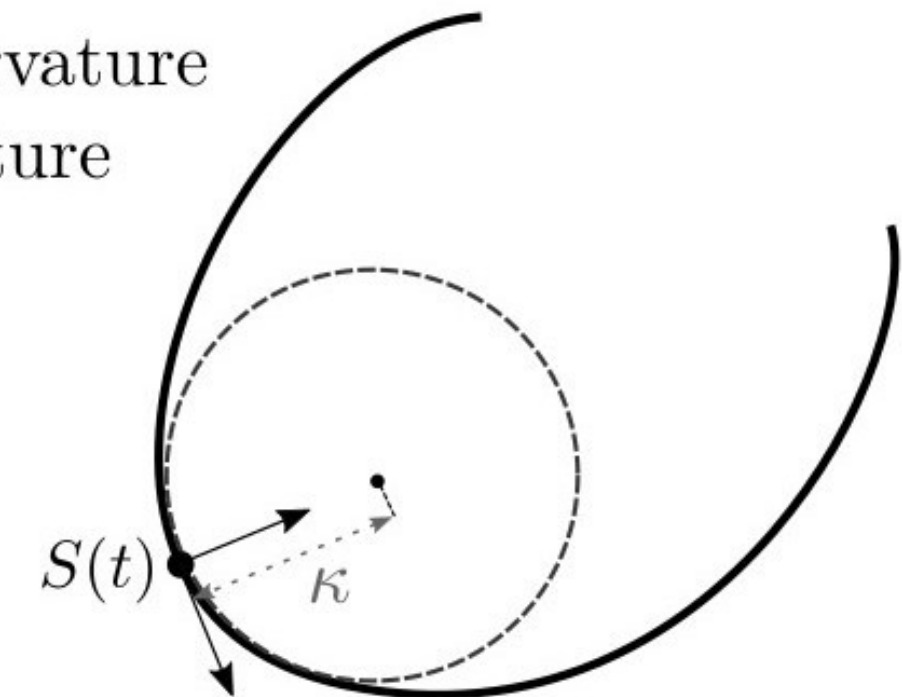
Curvature of a curve

κ is the radius of curvature

$\lambda = 1/\kappa$ is the curvature

$$\left| \begin{aligned} \kappa &= S'(\rho) \\ \rho &= \int \|S'(t)\| \end{aligned} \right.$$

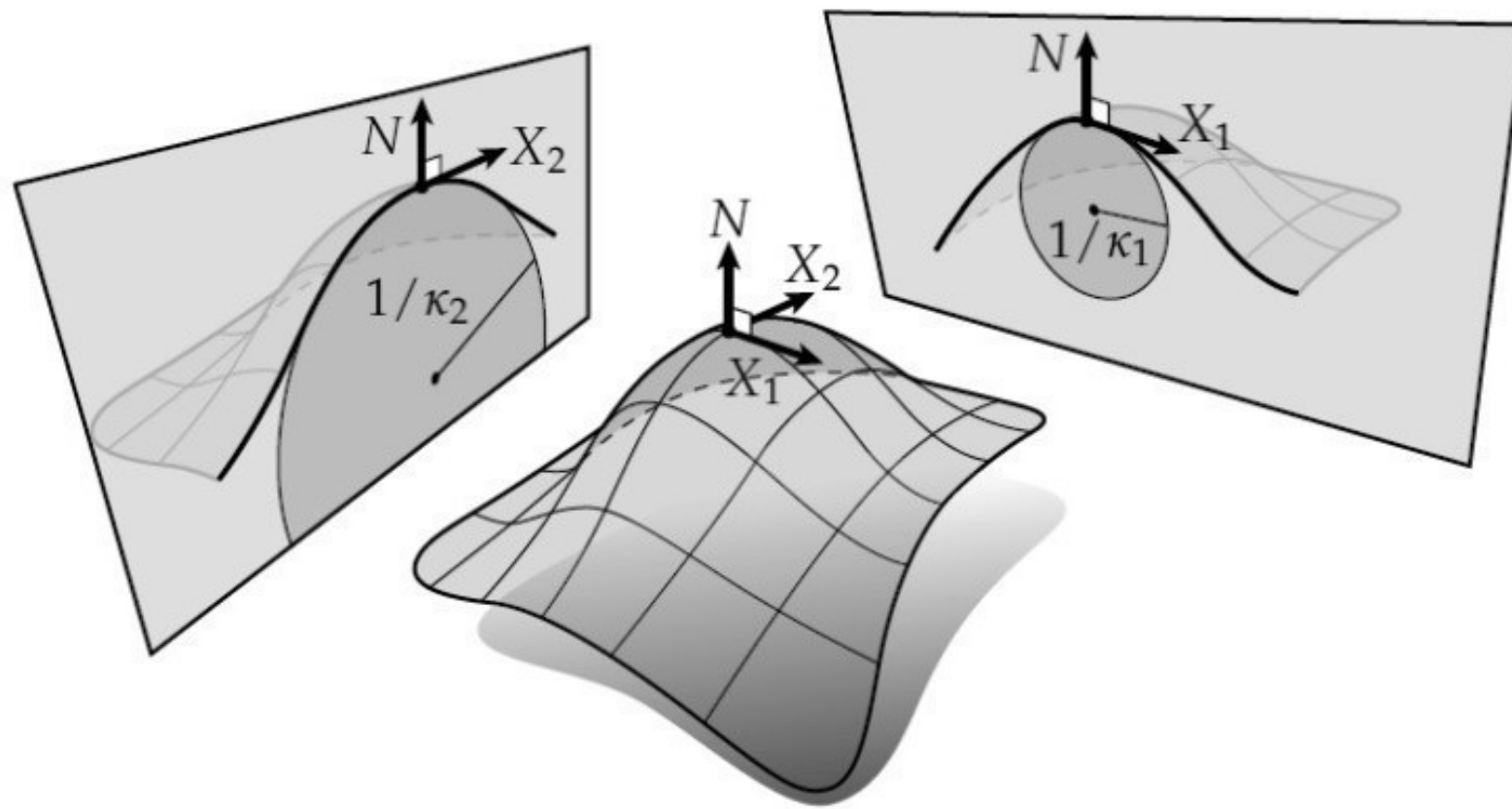
curvilinear parameterization



$$\kappa = \frac{1}{\|S'(t)\|^3} (\|S'(t)\|^2 \|S''(t)\|^2 - \langle S'(t), S''(t) \rangle^2)^{1/2}$$

007

Intuition for the curvature on a Surface



[Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus, SIGGRAPH 2013]

008

Tangent plane

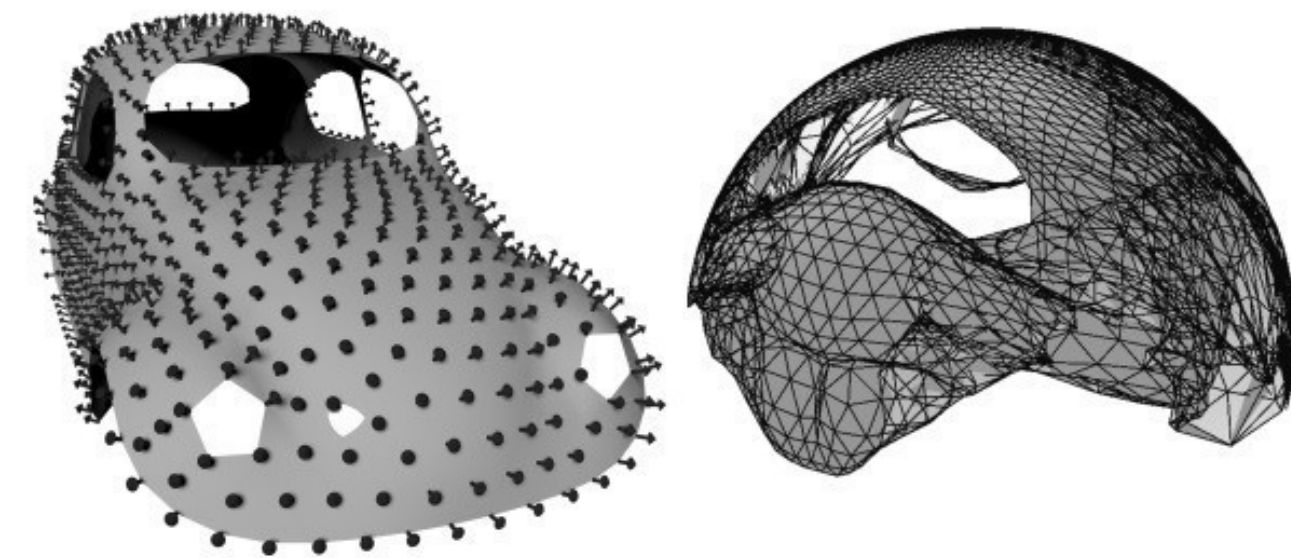
Surface in $\mathbb{R}^3$: $S(u, v) = (S_x(u, v), S_y(u, v), S_z(u, v))$

Normal: $n(u, v) = (S_u \times S_v) / \|S_u \times S_v\| \in \mathbb{S}^2$

Tangent space of S in $x_0 = S(u, v)$

$$T_{x_0}S = \text{Im}(D S(u, v)) = \{S_u(u, v) h_u + S_v(u, v) h_v | (h_u, h_v) \in \mathbb{R}^2\}$$

$$N: \begin{cases} \Gamma & \rightarrow \mathbb{S}^2 \\ x_0 = S(u, v) & \mapsto N(x_0) = n(u, v) \end{cases}$$

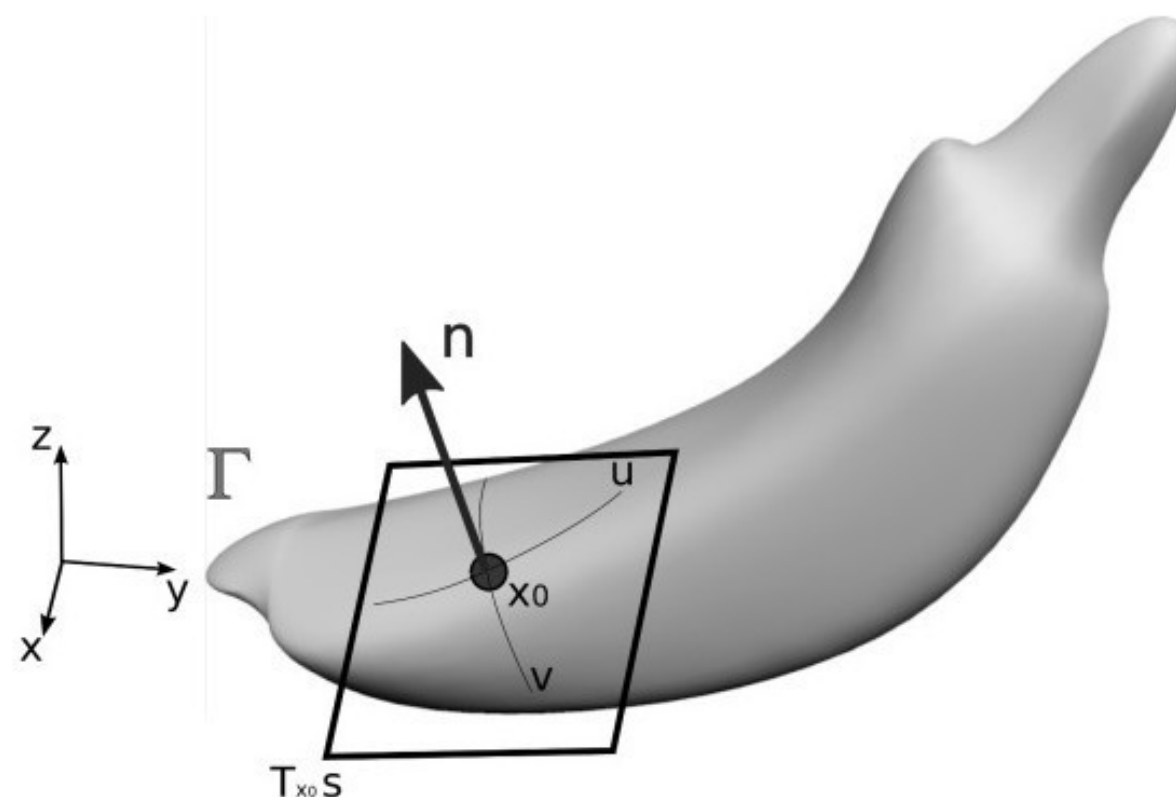


009

Integral properties

Area of Γ : $\iint_{(u,v) \in \mathcal{D}} \|S_u \times S_v\| \, du \, dv$

Volume defined by Γ $\iint_{(u,v) \in \mathcal{D}} S_z(u, v) n_z^S(u, v) \, du \, dv$



010

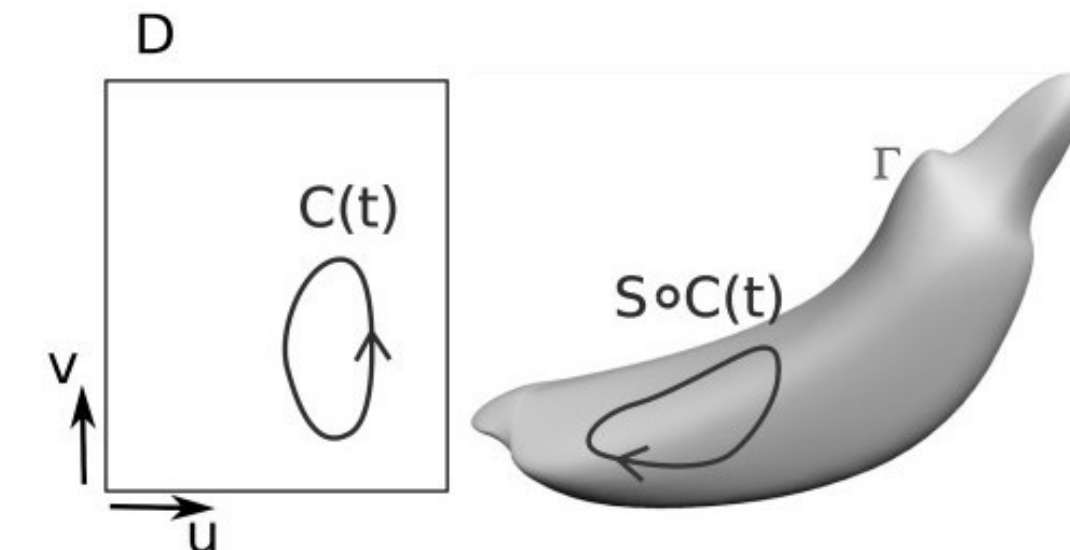
First fundamental form

Curve $C \subset \mathcal{D}$ of length $L = \int_t \langle C'(t), C'(t) \rangle^{1/2} \, dt$

Length of curve $C_s = S(C_x, C_y)$

$$L_s = \int_t \langle (S \circ C)'(t), (S \circ C)'(t) \rangle^{1/2} \, dt$$

$$= \int_t (C'^T(t) I_S(t) C'(t))^{1/2} \, dt$$



011

First fundamental form

Derivation $L_s = \int_t ((S \circ C)'^T(t) (S \circ C)'(t))^{1/2} dt$

$$(S \circ C)' = C'_x (S_{,u} \circ C) + C'_y (S_{,v} \circ C)$$

$$(S \circ C)' = (S_{,u} \ S_{,v}) \begin{pmatrix} C'_x \\ C'_y \end{pmatrix} \\ = \partial S^T C'$$

$$(S \circ C)'^T (S \circ C)' = (C'^T \partial S) (\partial S^T C') \\ = C'^T (\partial S \partial S^T) C' \\ = \boxed{C'^T I_S C'}$$

012

First fundamental form

I_S First fundamental form / metric tensor

$$I_S = \begin{pmatrix} S_{,u}^2 & \langle S_{,u}, S_{,v} \rangle \\ \langle S_{,u}, S_{,v} \rangle & S_{,v}^2 \end{pmatrix}$$

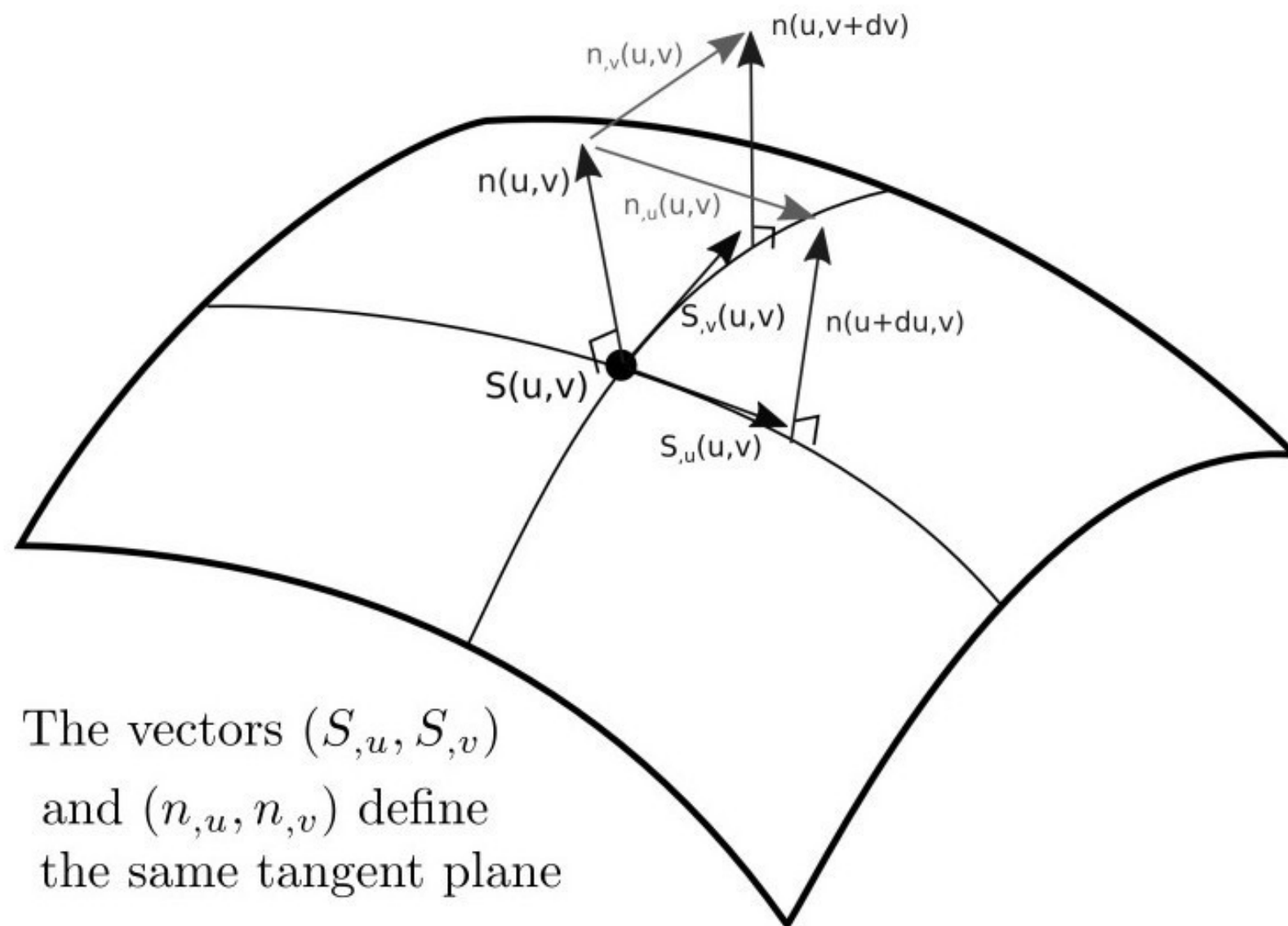
I_S quadratic form associated to $\langle dS, dS \rangle$

$\sqrt{\det(I_S)}$ = local area change

$$\text{Area of } \Gamma = \iint_{(u,v) \in \mathcal{D}} \sqrt{\det(I_S)} du dv$$

013

Derivative of the normals



The vectors $(S_{,u}, S_{,v})$
and $(n_{,u}, n_{,v})$ define
the same tangent plane

014

Second fundamental form

The vectors $(n_{,u}, n_{,v})$ can be expressed in the $(S_{,u}, S_{,v})$ plane

$$\begin{cases} n_{,u} = w_{00} S_{,u} + w_{01} S_{,v} \\ n_{,v} = w_{10} S_{,u} + w_{11} S_{,v} \end{cases} \quad \begin{pmatrix} n_{,u}^T \\ n_{,v}^T \end{pmatrix} = W_S \begin{pmatrix} S_{,u}^T \\ S_{,v}^T \end{pmatrix}$$

In multiplying both sides by $(S_{,u}, S_{,v})$

$$II_S = W_S I_S$$

Where II_S is the second fundamental form associated to S

$$II_S = \begin{pmatrix} \langle n_{,u}, S_{,u} \rangle & \langle n_{,u}, S_{,v} \rangle \\ \langle n_{,v}, S_{,u} \rangle & \langle n_{,v}, S_{,v} \rangle \end{pmatrix}$$

015

Relation with second derivatives

Using orthogonality

$$\begin{cases} \langle n, S_u \rangle = 0 \\ \langle n, S_v \rangle = 0 \end{cases}$$

Differentiating

$$\begin{cases} \langle n_u, S_u \rangle = -\langle n, S_{uu} \rangle \\ \langle n_v, S_u \rangle = -\langle n, S_{uv} \rangle \\ \langle n_v, S_v \rangle = -\langle n, S_{vv} \rangle \end{cases}$$

II can be expressed from n and the second derivatives of S

$$\Pi_S = - \begin{pmatrix} \langle n, S_{uu} \rangle & \langle n, S_{uv} \rangle \\ \langle n, S_{uv} \rangle & \langle n, S_{vv} \rangle \end{pmatrix}$$

016

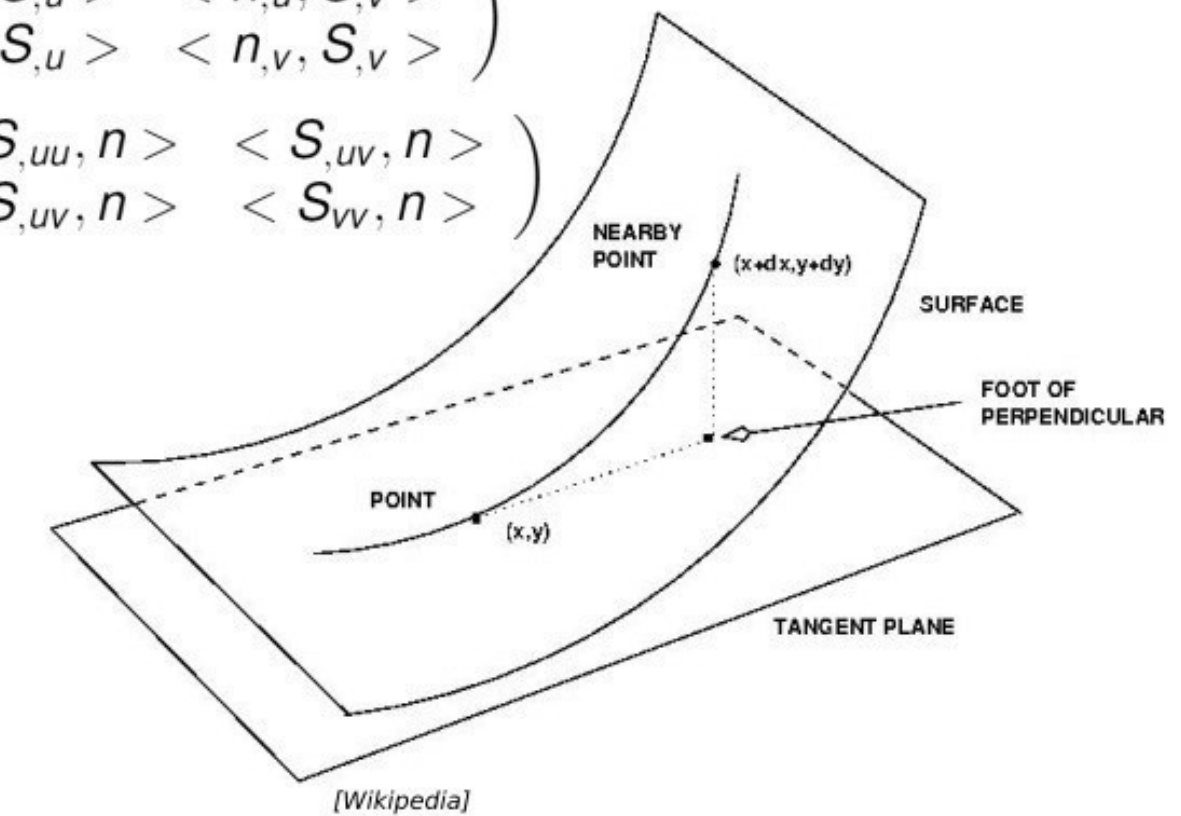
Second fundamental form

II is the quadratic form associated to $\langle sS, dn \rangle$

= Taylor expansion of S in its tangent plane

$$\Pi_S = \begin{pmatrix} \langle n_u, S_u \rangle & \langle n_u, S_v \rangle \\ \langle n_v, S_u \rangle & \langle n_v, S_v \rangle \end{pmatrix}$$

$$\Leftrightarrow \Pi_S = - \begin{pmatrix} \langle S_{uu}, n \rangle & \langle S_{uv}, n \rangle \\ \langle S_{uv}, n \rangle & \langle S_{vv}, n \rangle \end{pmatrix}$$



017

Weingarten application

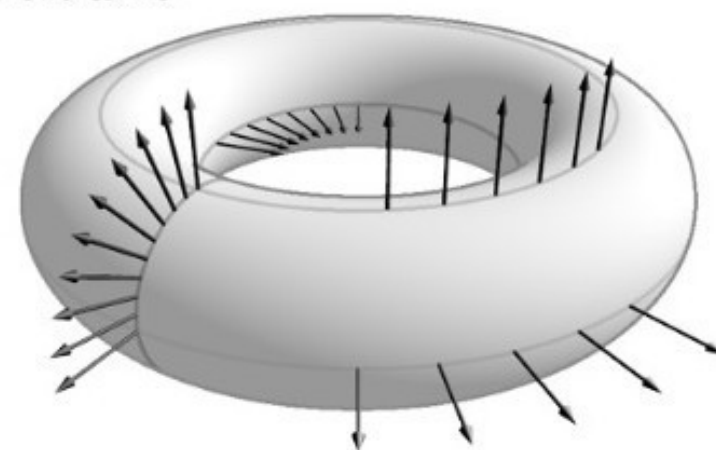
The matrix W_S such that $W_S = \Pi_S I_S^{-1}$ is the Weingarten matrix (or Shape operator)

W_S is diagonalizable and has real eigenvalues

$$W_S = V^T \Lambda V \quad \text{with} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

λ_1, λ_2 are the principal curvature

Weingarten map =
differential of the Gauss map



018

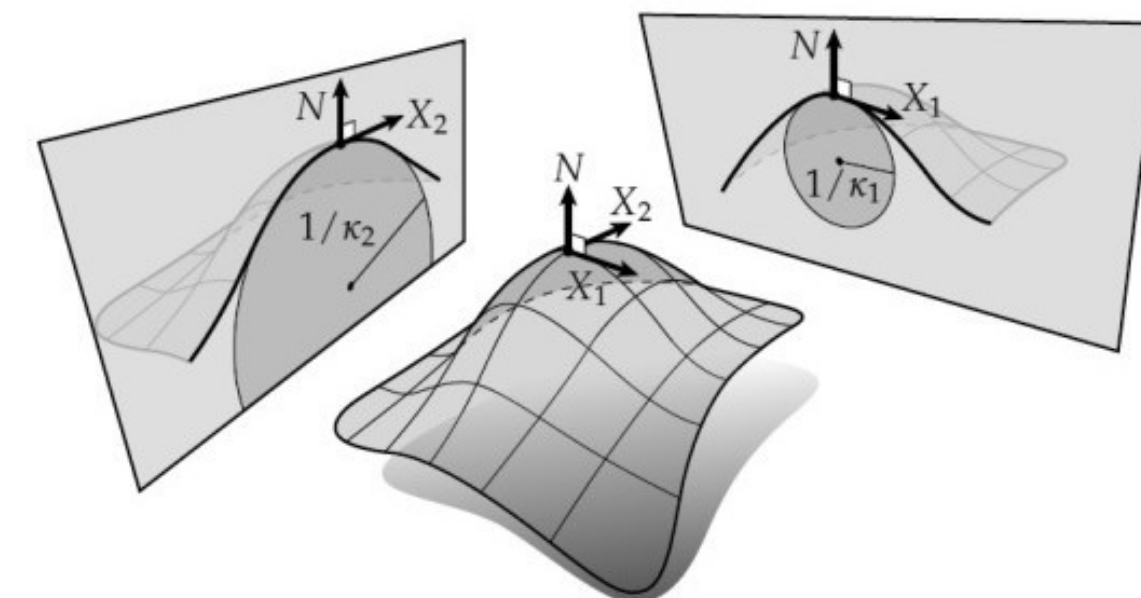
Principal curvatures

Eigenvalues of W_S : principal curvatures (λ_1, λ_2) .

Principal radius of curvatures $(\kappa_1 = \lambda_1^{-1}, \kappa_2 = \lambda_2^{-1})$.

Eigenvectors of W_S :

direction $(\mathbf{v}_1, \mathbf{v}_2)$ of the principal curvatures.



[Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus, SIGGRAPH 2013]

019

Curvature types

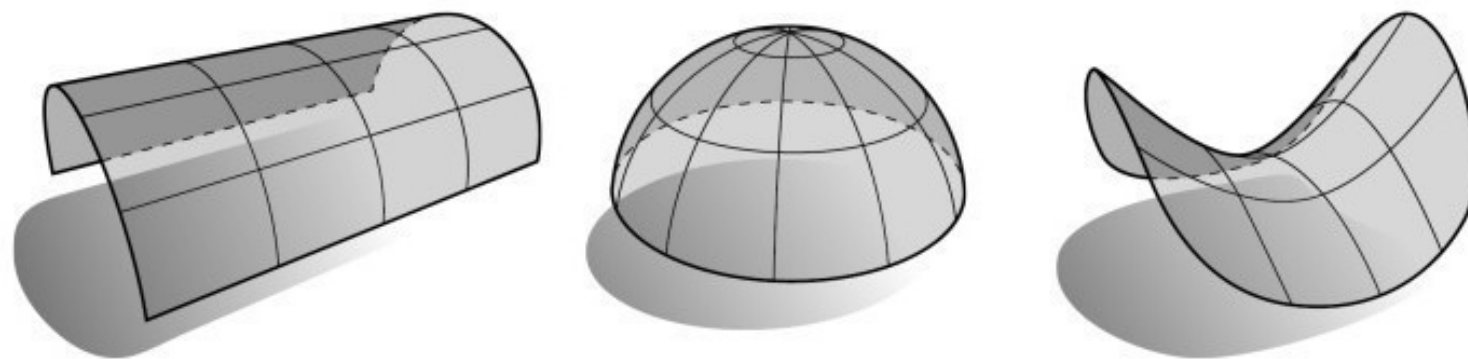
A surface can be locally

Planar: $\lambda_1 = \lambda_2 = 0$

Cylindrical: $\lambda_i \neq 0, \lambda_j = 0$

Elliptic: $\lambda_i \lambda_j > 0$

Hyperbolic: $\lambda_i \lambda_j < 0$



[Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus, SIGGRAPH 2013]

020

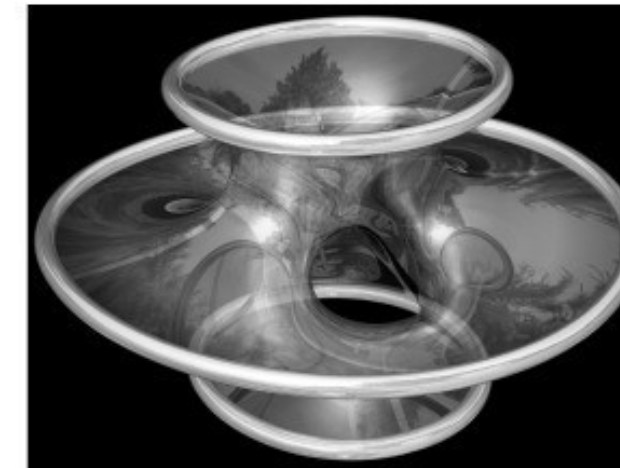
Gauss and mean curvatures

Gauss curvature: $K_s = \lambda_1 \lambda_2 = \det(W_S) = \frac{\det(II_S)}{\det(I_S)}$

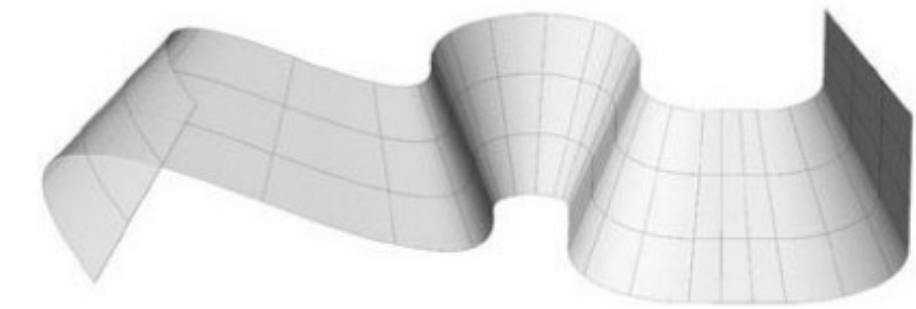
Mean curvature: $H_S = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\text{tr}(W_S)$

$H_S = 0 \Leftrightarrow S$ is a minimal surface

$K_S = 0 \Leftrightarrow S$ is a developable surface



[Paul Nylander]



[M. Nettelbladt]

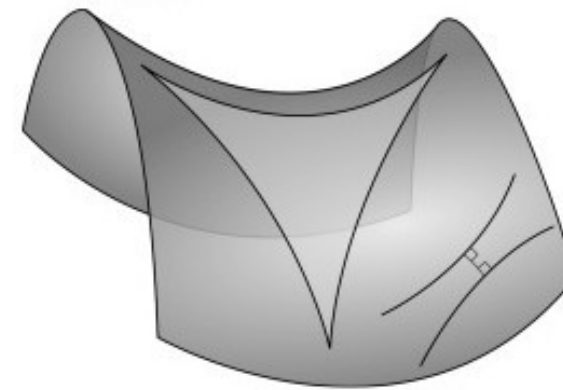
021

Integral relation

Gauss-Bonnet theorem: $\int_S K_s dA + \int_{\partial S} k_g ds = 2\pi\chi(S)$

k_g : geodesic curvature

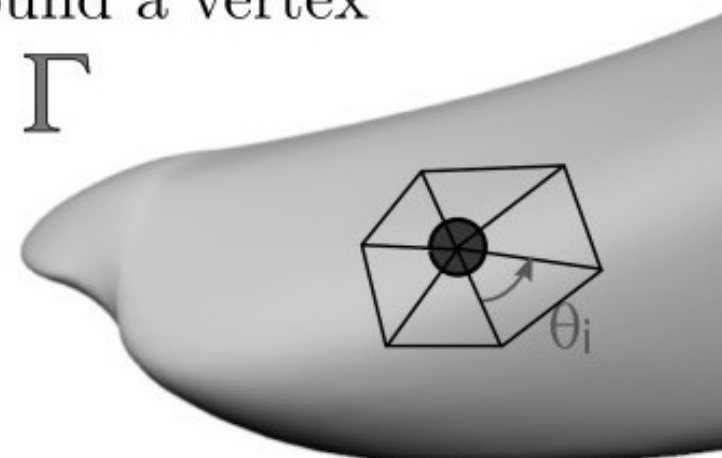
χ : Euler characteristic
(topological invariant)



[Wikipedia]

Application to a mesh around a vertex

$$K \simeq \frac{1}{A} \left(\sum_i \theta_i - 2\pi \right) \Gamma$$



022