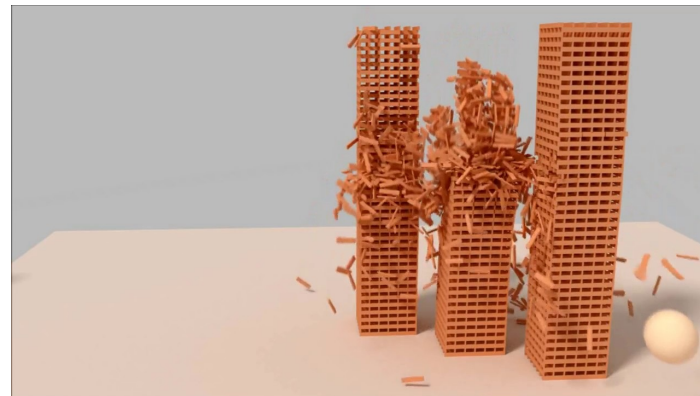
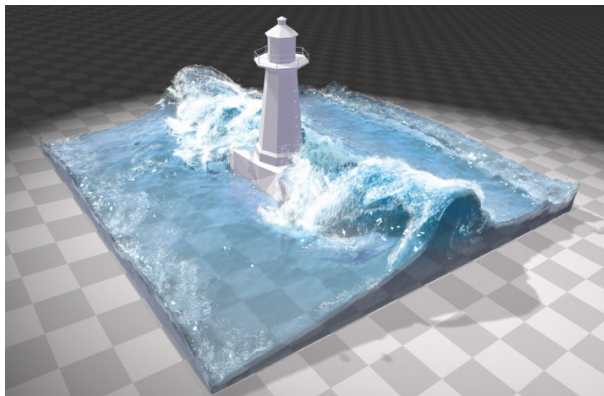


Physically based simulation

When physically based simulation is needed

- Accurate dynamics
- Tedious to model by hand or procedurally
 - Multiple interacting elements: ex. Multiple collisions: rigid bodies, hairs, etc.
 - Complex animated geometry: Cloths, fluids



General methodology

1. Description of the system - Describe system by some parameters (positions, speed, orientation, etc).

- State of the system is known at time $t = 0$ - *Initial value problem in time*
- State of the system may be constrained in space - *Boundary value problem in space*

2. Evolution Link the evolution of the system to forces or constraints using dynamic principles and conservation laws

⇒ Differential equation

3. Numerical Solution Solve the differential equation using numerical iterative approaches.

Note: Fundamentally different than direct approach controlling the trajectories at key-frames

- The system is set at an initial step
 - We let the numerical solution build the space-time trajectory for us
- (+) Allows to model complex behavior
- (-) Lack of control on the result

Physically based models

1- Particles

2- Rigid bodies

3- Continuum models

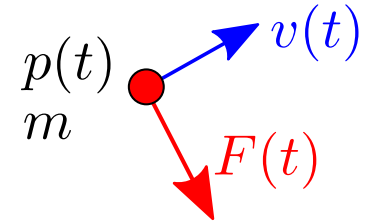
Physically-based particle system

1. Description

Particle is fully described by: Position p , Speed v , Mass m

fundamental quantities: position and linear momentum $P = m v$

fundamental quantity = preserved through time when no external forces are applied



2. Evolution

- Fundamental principle of the dynamics

Force applied on particle $F(p, v, t)$

$$\begin{cases} p'(t) = v(t) \\ P'(t) = m v'(t) = F(p, v, t) \end{cases}$$

- Conservation of energy (ex. kinetic energy $(1/2 m v^2)$ +potential energy = const, etc.)

- Lagrangian, or Hamiltonian

Physically-based particle system

3. Numerical Solution

ODE (Ordinary Differential Equation) formulated as an **Initial Value Problem**

$$\text{ex. } \begin{cases} p'(t) = v(t) \\ mv'(t) = F(p, v, t) \end{cases}, \quad \text{with } v(0) = v_0, p(0) = p_0$$

- Discretize in time $t^k = k h$, $h = \Delta t = \text{time step}$.

$\Rightarrow$ Build a discrete numerical solution $p^k = p(t^k)$, $v^k = v(t^k)$.

- We can consider initially the following iterative scheme

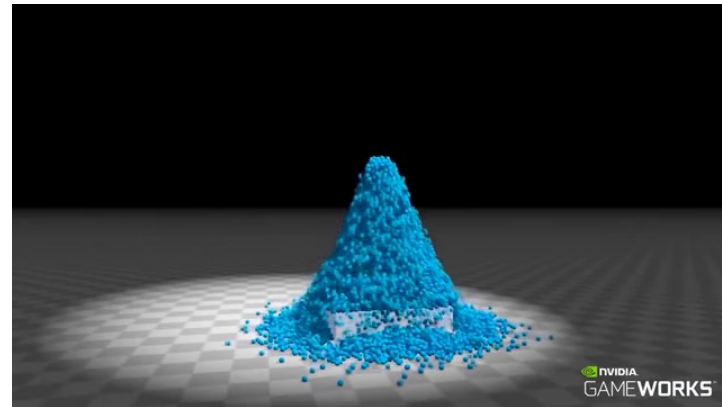
$$\begin{cases} v^{k+1} = v^k + h F(p^k, v^k, t^k) \\ p^{k+1} = p^k + h v^{k+1} \end{cases}$$

Simple to implement, reasonably OK for simple examples (more details later).

Physically-based particle system

Pro

- (+) Simple to implement, and control.
- (+) Efficient to compute, scalable
- (+) Highly adaptable from simple particle to rigid and deformable models



Cons

- (-) Limited accuracy - highly simplified model from physical point of view.
- ⇒ Dominant model in CG production for general purpose deformable model animation.

Common use: Lots of sparsely interacting particles

Physically based models

1- Particles

2- Rigid bodies

3- Continuum models

Orientation and angular speed

In addition of position and speed, rigid body is defined by its **orientation** r and **angular speed** ω .

- In matrix form $r \in \mathbb{R}^{3 \times 3}$

- Angular speed $\omega = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$

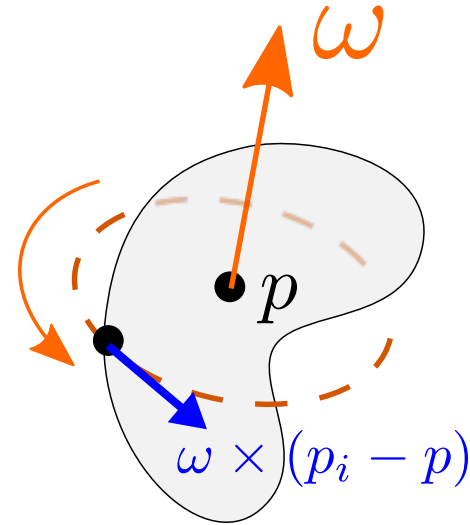
Every vector $p_i(t)$ of the rigid body has a speed $p'_i(t) = v(t) + \omega(t) \times (p_i(t) - p(t))$

- Matrix expression of the angular speed

$$M_\omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

- Derivative of orientation matrix

$$r'(t) = M_\omega(t) r(t)$$



Rigid bodies system description

Similar to particle: Position $p \in \mathbb{R}^3$ and speed $v \in \mathbb{R}^3$ of the center of mass. Mass of the solid $m \in \mathbb{R}$.

- $p'(t) = v(t)$
- Fundamental quantity: position p and linear momentum $P = m v$.

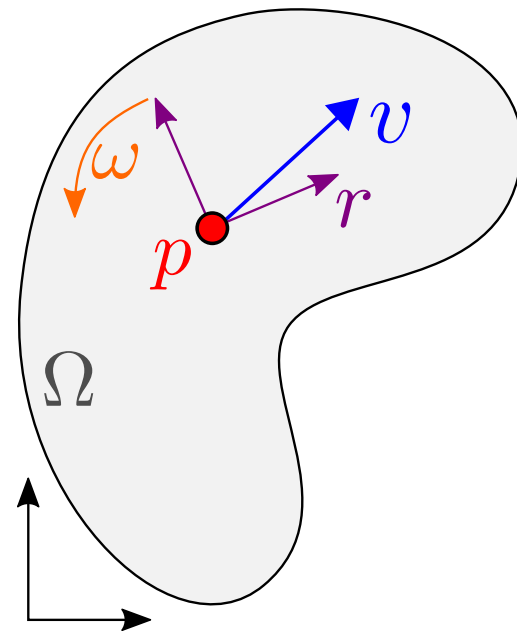
Addition of rigid body: **Orientation** $r \in \mathbb{R}^{3 \times 3}$ (in its matrix formulation), and **angular speed** $\omega \in \mathbb{R}^3$ of the solid.

- $r'(t) = M_\omega(t) r(t)$
- Fundamental quantity: orientation r and **angular momentum**

$L = I \omega \in \mathbb{R}^3$, with $I \in \mathbb{R}^{3 \times 3}$ **inertia tensor** of the solid.

Intuition

- Solid acts like a particle at its center of mass + orientation and angular speed
- Mass $\rightarrow$ resistance to change of linear speed
- Inertia tensor ($\simeq$ angular mass) $\rightarrow$ resistance to change of angular speed



Definition of quantities

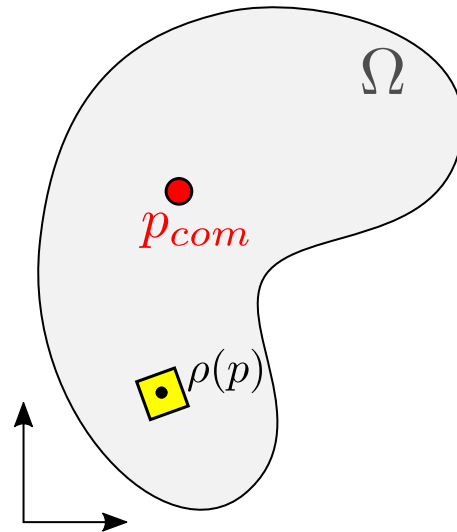
- Solid defined within a domain $\Omega \subset \mathbb{R}^3$
- With a density of mass $\rho(p)$ at each point $p \in \Omega$

- Total mass of the solid m

$$m = \int_{p \in \Omega} \rho(p) \, d\Omega$$

- Position of the center of mass p_{com}

$$p_{com} = \frac{1}{m} \int_{p \in \Omega} \rho(p) p \, d\Omega$$



Definition of inertia

Inertia tensor expressed at point q

Calling r the local coordinate $r = p - q$

$$I = \int_{r \in \Omega} \rho(r) \begin{pmatrix} I_{xx}(r) & I_{xy}(r) & I_{xz}(r) \\ I_{xy}(r) & I_{yy}(r) & I_{yz}(r) \\ I_{xz}(r) & I_{yz}(r) & I_{zz}(r) \end{pmatrix} d\Omega = \int_{r \in \Omega} \rho(r) \begin{pmatrix} r_y^2 + r_z^2 & -r_x r_y & -r_x r_z \\ -r_x r_y & r_x^2 + r_z^2 & -r_y r_z \\ -r_x r_z & -r_y r_z & r_x^2 + r_y^2 \end{pmatrix} d\Omega$$

$$I = \int_{r \in \Omega} \rho(r) (r^T r \text{Id} - r r^T) d\Omega$$

Rem.

- I is usually expressed at the center of mass $q = p_{com}$
- I depends on the body orientation. Given a rotation r : $I = r I_0 r^T$
 $\Rightarrow$ compute once I_0 in a rest position, then update it using r
- There exist a frame in which I is diagonal (principle axes of inertia). Corresponds to eigenvectors of matrix I .

Forces and torques on a solid

- Given a local force $f(p)$ acting on a position $p \in \Omega$
- Contribute to 2 global components applied on the body:
 - Total **external force** F applied on the center of mass

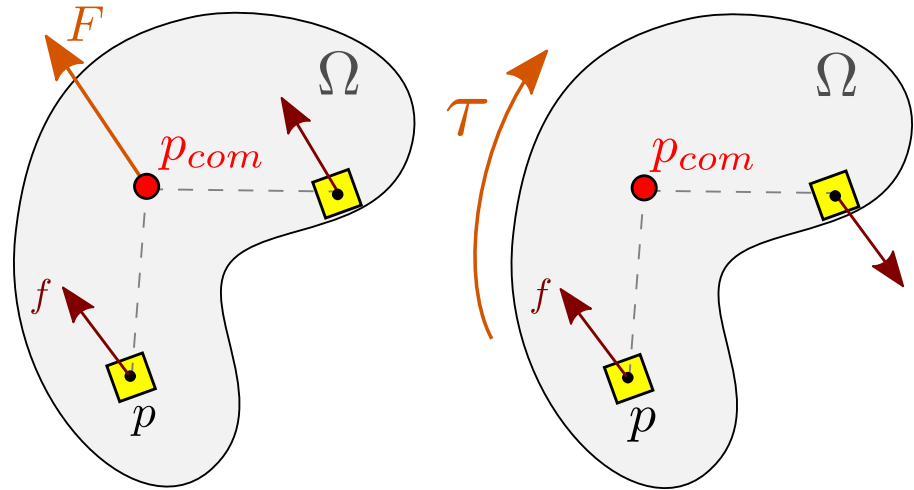
Induce linear displacement of COM

$$F = \int_{p \in \Omega} f(p) dp$$

- **Torque** τ applied on the body

Induce spinning of the solid around its COM

$$\tau = \int_{p \in \Omega} (p - p_{com}) \times f(p) d\Omega$$



Dynamics

Similarly to particles

Force F is related to the change of linear momentum $F(t) = P'(t) = (m v)'(t)$

Specific to rigid bodies

Torque τ is related to the change of angular momentum $\tau(t) = L'(t) = (I \omega)'(t)$

Equation of Motion

Fundamental principle of dynamics for rigid body

$$\frac{d}{dt} \begin{pmatrix} p \\ P \\ r \\ L \end{pmatrix} (t) = \begin{pmatrix} v(t) \\ F(t) \\ \omega(t) r(t) \\ \tau(t) \end{pmatrix}$$

Summary: Rigid solid dynamics

1. Description

State vector $u(t) = (p(t), P(t) = m v(t), r(t), L(t) = I(t) \omega(t))$

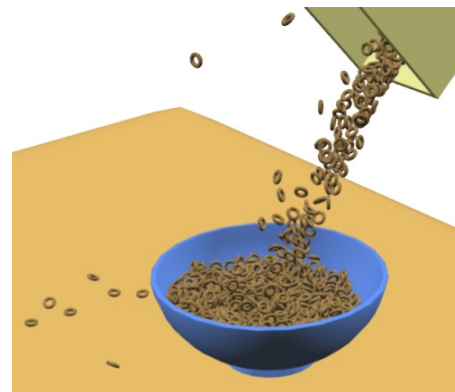
Initial condition $u(0) = (p_0, m v_0, r_0, I_0 \omega_0)$

2. Temporal Evolution

$$\frac{d}{dt} \begin{pmatrix} p \\ P \\ r \\ L \end{pmatrix} (t) = \begin{pmatrix} v(t) \\ F(t) \\ \omega(t) r(t) \\ \tau(t) \end{pmatrix}$$

- $I(t) = r(t) I_0 r(t)^T$

- $\omega(t) = I^{-1}(t) L(t)$



In practice:

- Shape and density ρ are known
- Compute I_0 for the given shape
- p_0, v_0, r_0, ω_0 are Initial Conditions

Iterate over t^k

- Compute $F(t^k)$ and $\tau(t^k)$ given external forces
- Compute $I(t^k) = r(t^k) I_0 r(t^k)^T$
- Compute $\omega(t^k) = I^{-1}(t^k) L(t^k)$
- Update state vector $p^{k+1}, P^{k+1}, r^{k+1}, L^{k+1}$ at t^{k+1}

Use of quaternion

Relation $r'(t) = \omega(t) r(t)$ leads in numerical drift from rotation matrix

$$\text{ex. } r^{k+1} = (\text{Id} + h \omega^k) r^k \quad (\text{explicit scheme})$$

Using quaternion leads to more robust behavior

- Quaternion expression: $q'(t) = \frac{1}{2} q_\omega(t) q(t)$, with $q_\omega(t) = (\omega(t), 0)$

- Quaternion is forced to keep a unit norm

$$\text{ex. } \begin{cases} q^{k+1} = q^k + \frac{1}{2} q_\omega^k q^k \\ q^{k+1} = q^{k+1} / \|q^{k+1}\| \end{cases}$$

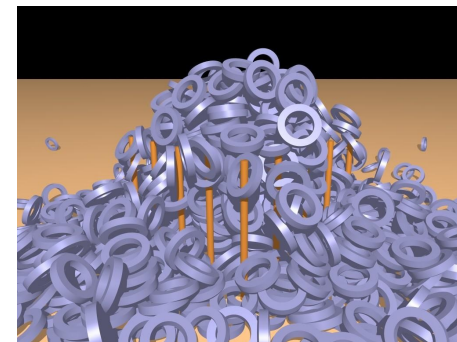
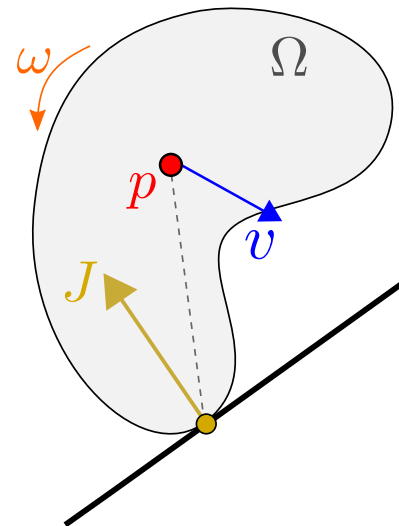
Collisions in rigid bodies

Collisions at position p change both linear and angular velocity

Similarly to particle: use of impulse (sudden change of speed) J .

Impulse split into

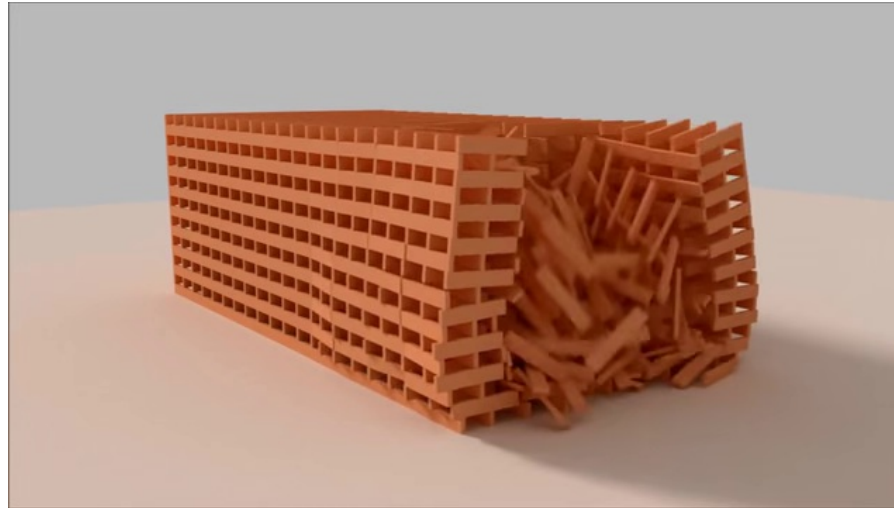
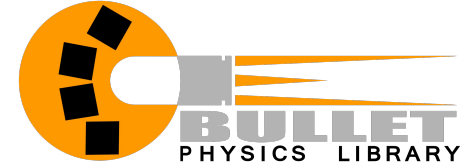
- Force impulse $\Delta P = m \Delta v = J$
- Torque impulse $\Delta L = I \Delta \omega = (p - p_{com}) \times J$



More details [D. Barff. Physically Based Modeling. SIGGRAPH Course Notes 1999]

Rigid bodies usage

- Standard usage for rigid bodies motions
 - Limited to non-deformable shapes
- Common in VFX (explosions), and *simulation games* (cars, airplanes, etc).
- Standard library: [Bullet physics](#) (ex. used in Blender).



Physically based models

1- Particles

2- Rigid bodies

3- Continuum models

Deformation of a continuous shape

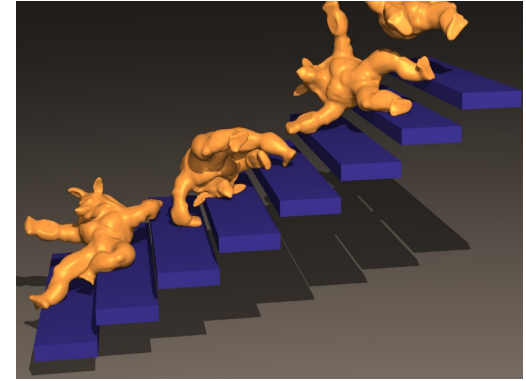
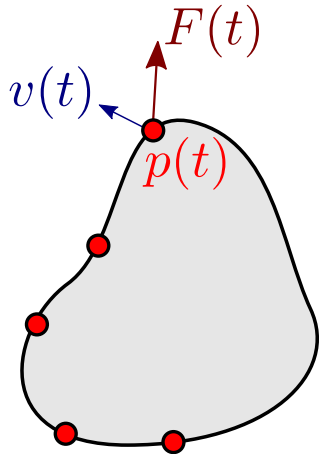
Every part of the shape can be deformed

ex. Describing elastic shapes, visco-elastic shapes, fluids, etc.

Two ways to describe the deforming object

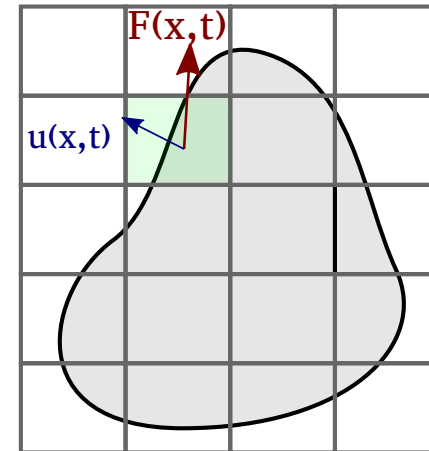
1. Lagrange representation

Positions follow the object deformation



2. Euler representation

Positions are fixed in 3D space



Deformation the Lagrangian description

Deformation map $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $p = \varphi(P)$

P position in the reference undeformed shape

p position in the deformed configuration.

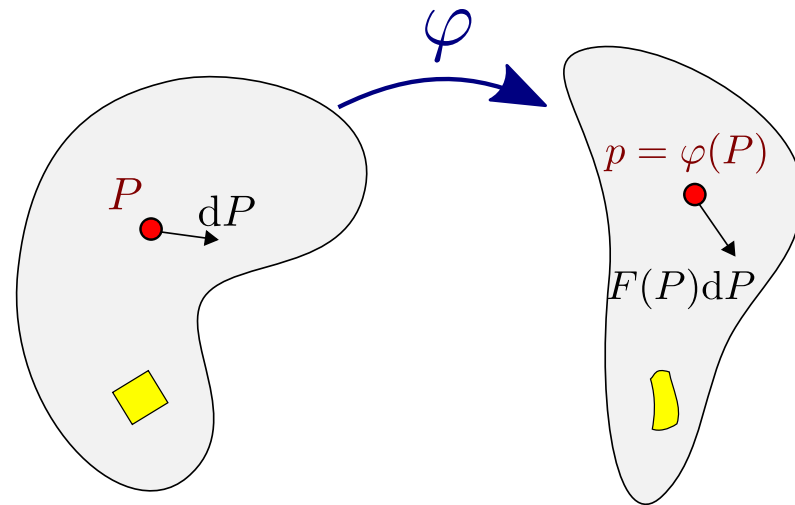
Deformation Gradient F

$$- F(P) = \frac{\partial \varphi}{\partial P}(P) = \frac{\partial p}{\partial P} \in \mathbb{R}^{3 \times 3}$$

- Characterizes the local deformation associated to φ

Position $P + dP$ is mapped into $\varphi(P + dP) \simeq p + \frac{\partial \varphi}{\partial P} dP$

$$- F(P) = \begin{pmatrix} \frac{\partial \varphi_x}{\partial X} & \frac{\partial \varphi_x}{\partial Y} & \frac{\partial \varphi_x}{\partial Z} \\ \frac{\partial \varphi_y}{\partial X} & \frac{\partial \varphi_y}{\partial Y} & \frac{\partial \varphi_y}{\partial Z} \\ \frac{\partial \varphi_z}{\partial X} & \frac{\partial \varphi_z}{\partial Y} & \frac{\partial \varphi_z}{\partial Z} \end{pmatrix}$$



Strain

Deformation gradient F describe both

- Rigid transformation (translation, rotation) - not related to material effort
- Any other deformation inducing local length change - related to material effort

Strain ϵ is a measure of deformation ignoring rigid transformation.

Several possible measure of strain

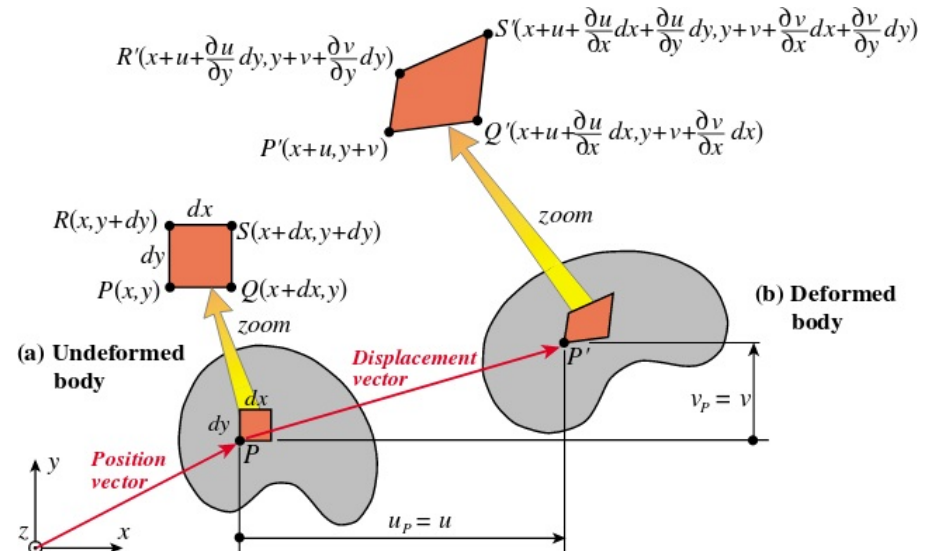
- Green strain tensor $\epsilon = \frac{1}{2} (F F^T - \text{Id})$

(+) If φ is a rotation $F = R \Rightarrow \epsilon = 0$

(-) Non linear in p

- Linearized Cauchy strain $\epsilon = \frac{1}{2} (F^T + F) - \text{Id}$

Used for small deformations



Stress

Stress $\sigma \in \mathbb{R}^{3 \times 3}$ describes internal forces (per area unit) induced by the local deformation (strain) in any direction

Constitutive Relation: Relation between stress and strain, characterize a type of material.

In its general form $\sigma_{ij} = \sum_{k,l} C_{ijkl} \epsilon_{kl}$, C : stiffness tensor (81 coefficients)

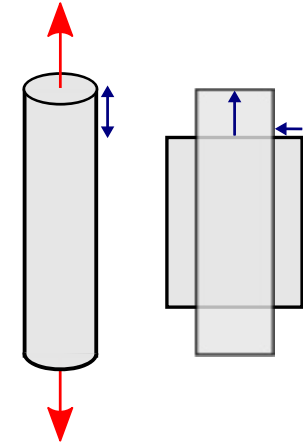
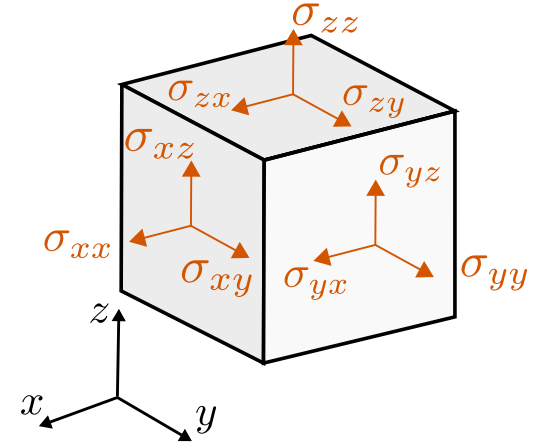
- Strain energy $U(\epsilon) = \sum_{i,j,k,l} C_{ijkl} \epsilon_{ij} \epsilon_{kl} / 2$

For homogeneous isotropic elastic material, constitutive relation simplifies to

$$\sigma = 2\mu \epsilon + \lambda \text{tr}(\epsilon) \text{Id}, (\mu, \lambda): \text{Lamé parameters}$$

Related to common mechanical modulus : Young' modulus Y and Poisson's ratio ν

$$\mu = \frac{Y}{2(1+\nu)}, \lambda = \frac{Y\nu}{(1+\nu)(1-2\nu)}$$



Evolution equation

Fundamental principle of dynamics in the entire volume Ω

Change of momentum = External forces (in volume) + Traction (stress applied on exterior surface normals)

$$\Rightarrow \underbrace{\int_{\Omega} \rho p''(t) d\Omega}_{\text{Change of momentum}} = \underbrace{\int_{\Omega} F(t) d\Omega}_{\text{External forces}} + \underbrace{\int_{\partial\Omega} \sigma n dS}_{\text{Traction force on the boundary}}$$

Using divergence theorem $\int_{\partial\Omega} \sigma n dS = \int_{\Omega} \text{div}(\sigma) d\Omega$

Equation in volume satisfied at each position $p \in \Omega$

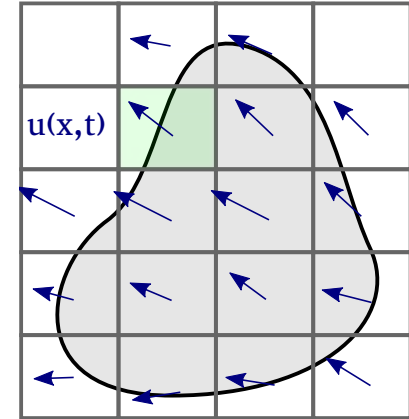
$$\boxed{\rho p''(t) = F(t) + \text{div}(\sigma(t))} \quad \sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \quad \text{div}(\sigma) = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

Euler formulation

In Euler formulation quantities are expressed at fixed position in 3D space.

Deformation described by velocity $u(p, t)$ at a given 3D fixed point $p = (x, y, z)$ at time t .

- Do not require anymore a reference shape
- Usefull for heavily deforming shapes (ex. fluids, gaz).
- Change of speed during dt
$$\frac{du}{dt}(p, t) = \frac{\partial u}{\partial t} + \frac{\partial p}{\partial t} \cdot \frac{\partial u}{\partial p} = \frac{\partial u}{\partial t} + u \cdot \nabla u. \quad \text{Called } \textit{material derivative}.$$
- Similarly to Lagrangian derivation:
 - Stress-rate tensor ϵ (rate of change of deformation in a neighborhood of a point) expressed with respect to u
$$\epsilon = \frac{1}{2} (\nabla u + \nabla u^T)$$
 - Strain-rate tensor σ (rate of change of deformation in a neighborhood of a point).



Equation of motion for a fluid

- Fundamental principle of dynamics on linear momentum

$$\rho \frac{du}{dt} = F + \operatorname{div}(\sigma)$$

$$\Rightarrow \rho \frac{\partial u}{\partial t} = F + \operatorname{div}(\sigma) - u \cdot \nabla u. \quad \text{The term } u \cdot \nabla u \text{ is called } \textit{convection}.$$

- External force: weight $F = \rho g$

- Stress decomposed into

$$\sigma = \sigma_{viscous} + \sigma_{pressure}$$

$$\sigma_{pressure} = -p \operatorname{Id} \text{ (pressure acts along normal of surface elements)}$$

$$\rho \frac{\partial u}{\partial t} = \rho g - u \cdot \nabla u + \operatorname{div}(\sigma_{viscous} - p \operatorname{Id})$$

$$\Rightarrow \rho \frac{\partial u}{\partial t} = \rho g - u \cdot \nabla u - \nabla p + \operatorname{div}(\sigma_{viscous})$$

Navier-Stokes equation

- Newtonian fluid $\Rightarrow$ Linear relation between strain-rate ϵ and stress-rate $\sigma_{viscous}$
 - $\sigma_{viscous} = 2\mu \epsilon = \mu (\nabla u + \nabla u^T)$, μ constant viscosity parameter
- Incompressible fluid $\Rightarrow \text{div}(u) = 0$

Equation of motion

$$\Rightarrow \rho \frac{\partial u}{\partial t} = \rho g - u \cdot \nabla u - \nabla p + \text{div}(\mu (\nabla u + \nabla u^T))$$

- Noting that $\text{div}(\nabla u^T) = \nabla \text{div}(u) = 0$
- And $\text{div}(\nabla u) = \Delta u$

$$\Rightarrow \rho \frac{\partial u}{\partial t} = \rho g - u \cdot \nabla u - \nabla p + \mu \Delta u$$

Navier-Stokes equation for incompressible Newtonian fluid.

Numerical solutions

Lagrangian and Euler description leads to PDE (Partial Differential Equations)

In general: no explicit solutions $\Rightarrow$ approximate numerical solution

Finite Differences (FD)

- Discretize in space on a grid Δx and time Δt
- Use numerical approximation of derivatives using masks in space and time
 - (+) Very general, simple to setup
 - (+) Works well with rectangular grid (ex. Euler description)
 - (-) Difficult to handle shape boundaries
 - (-) Instabilities

Finite elements method (FEM)

- Discretize the shape into simple elements. Build continuous function on each element.
 - In CG: Elements are tetrahedron (in volume). Continuous function are barycentric coordinates (linear interpolation functions).
- Integrate PDE over each element (weak formulation), leads to a linear system
 - (+) Handle boundaries (ex. Deforming solid)
 - (+) Guarantee on accuracy
 - (-) Complex to set up, and computationally heavy
 - (-) Requires good quality meshing

