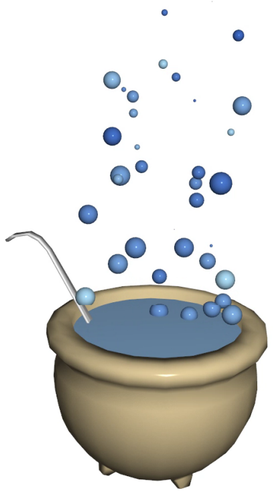
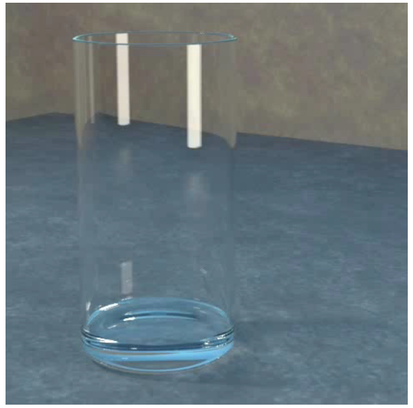
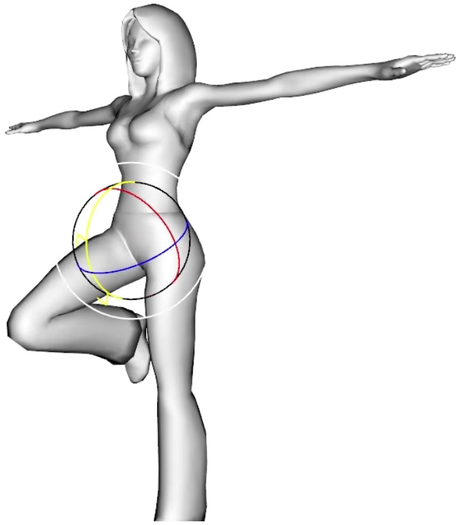
Animation 3D







Damien Rohmer

Damien.Rohmer@polytechnique.edu

EPITA Majeure Image

02/12/2019

Présentation générale

1. Présentation: Enseignant, groupe de recherche
2. Le cours "Animation 3D": Plan, évaluation
3. Rappels en Informatique Graphique

Geometric & Visual Computing - Présentation

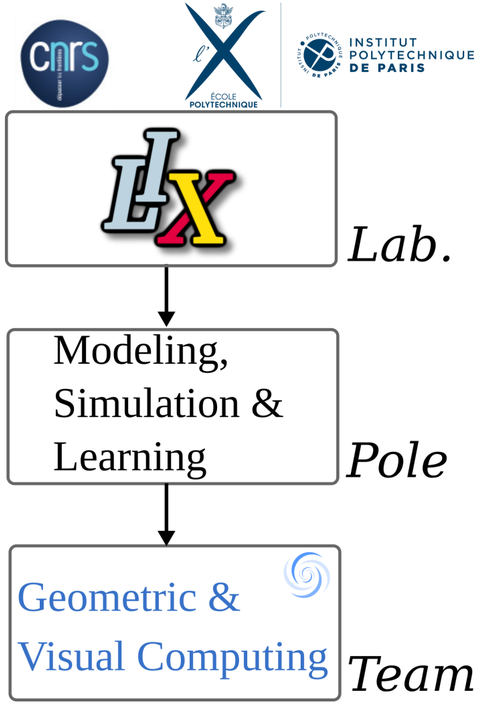
Equipe **Informatique Graphique** et **Vision**

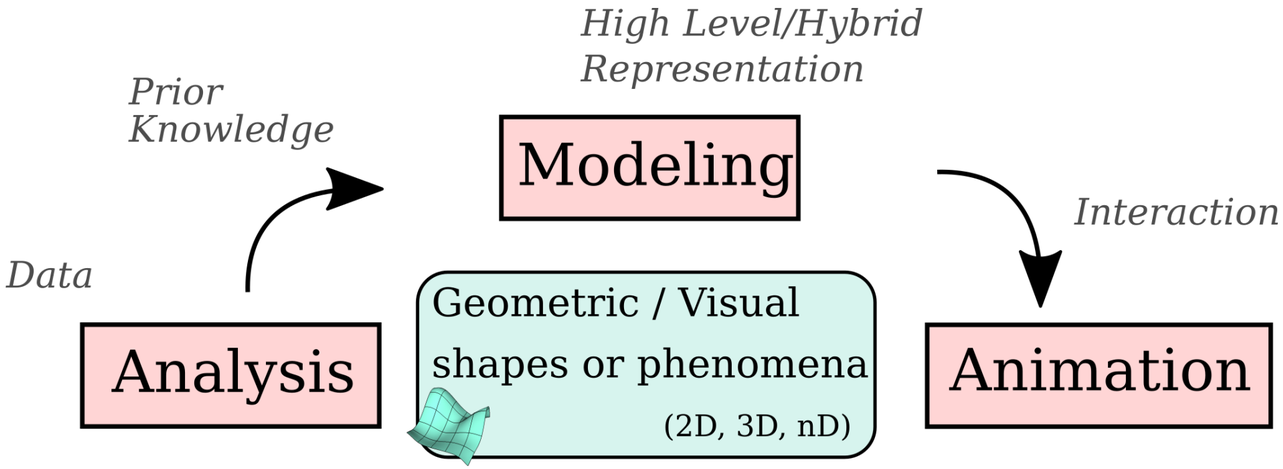
Recherche en **analyse géométrique**, **modélisation** et **animation**.

www.lix.polytechnique.fr/geovic/

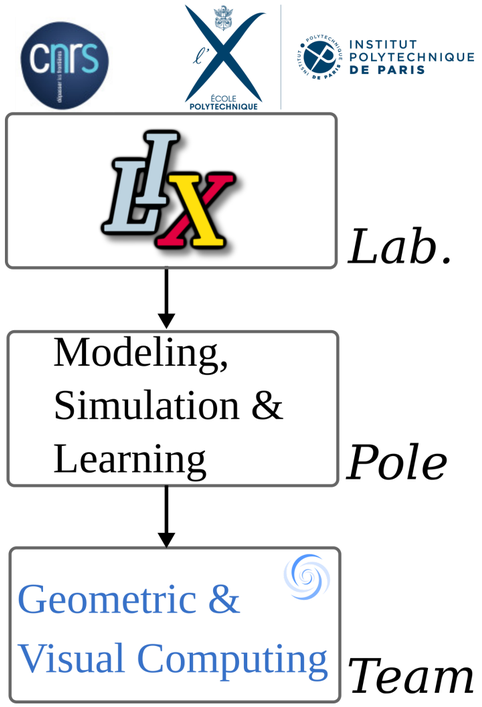
Partie du **LIX** (Laboratoire d'Informatique de l'Ecole Polytechnique), CNRS/X, IP Paris

Pole "**Modélisation, Simulation & Apprentissage**"





Outils sous-jacents: Geométrie algorithmique, Apprentissage, Optimisation, Modélisation mathématique, etc.

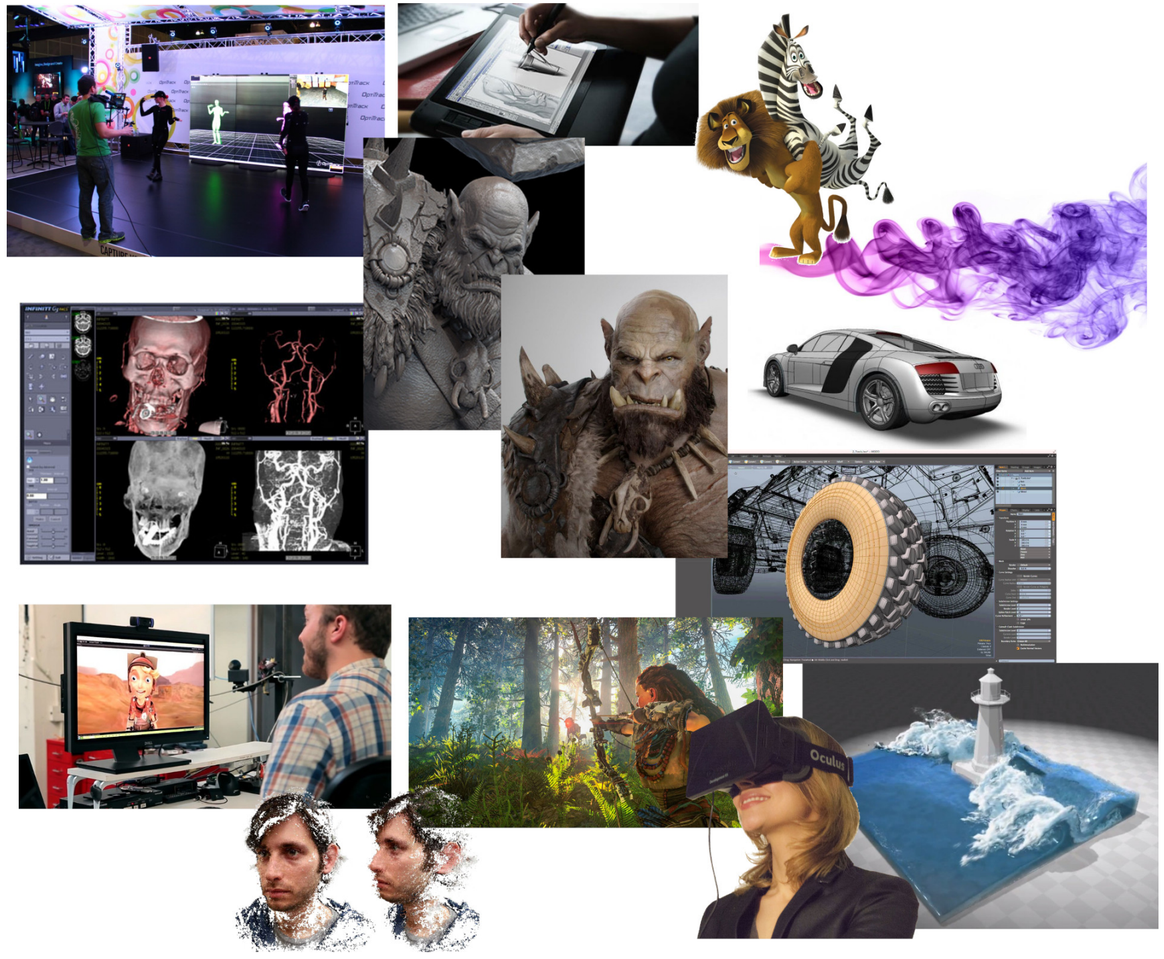


Contexte & challenges

Omniprésence des **modèles virtuels**

> Augmentation constante de la demande pour de l'**analyse** et de la **création** de contenu virtuel.

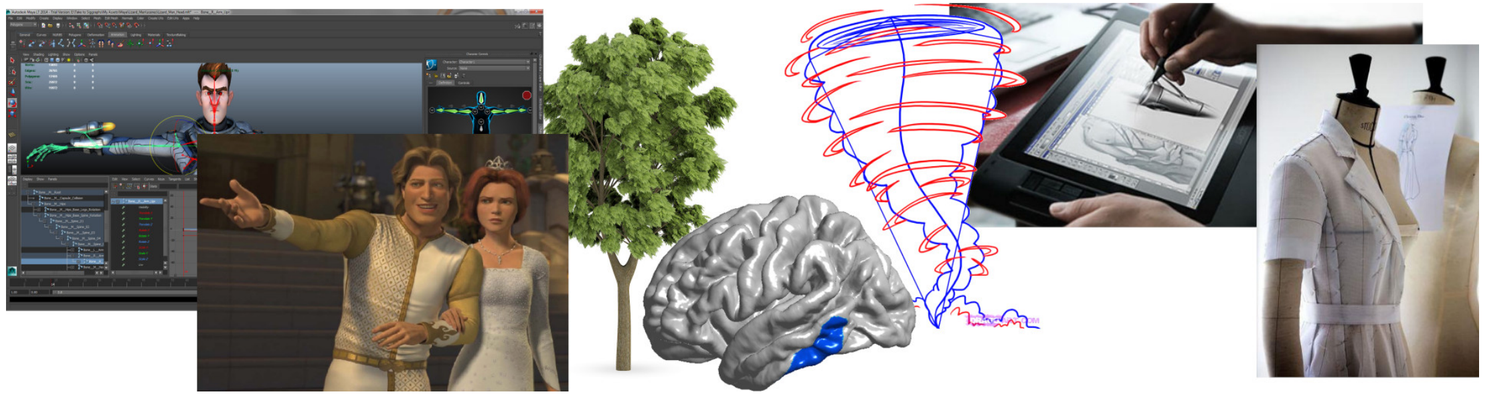
- Données hétérogènes, objectifs, usages
- Complexes, incomplètes, grands volumes



> Comment **Interpréter, calculer, modéliser & éditer** ces contenus **Geometriques & Visuels**

> Développer algorithmes et méthodes pour améliorer l'efficacité, la précision, l'interactivité et le controle.

Applications

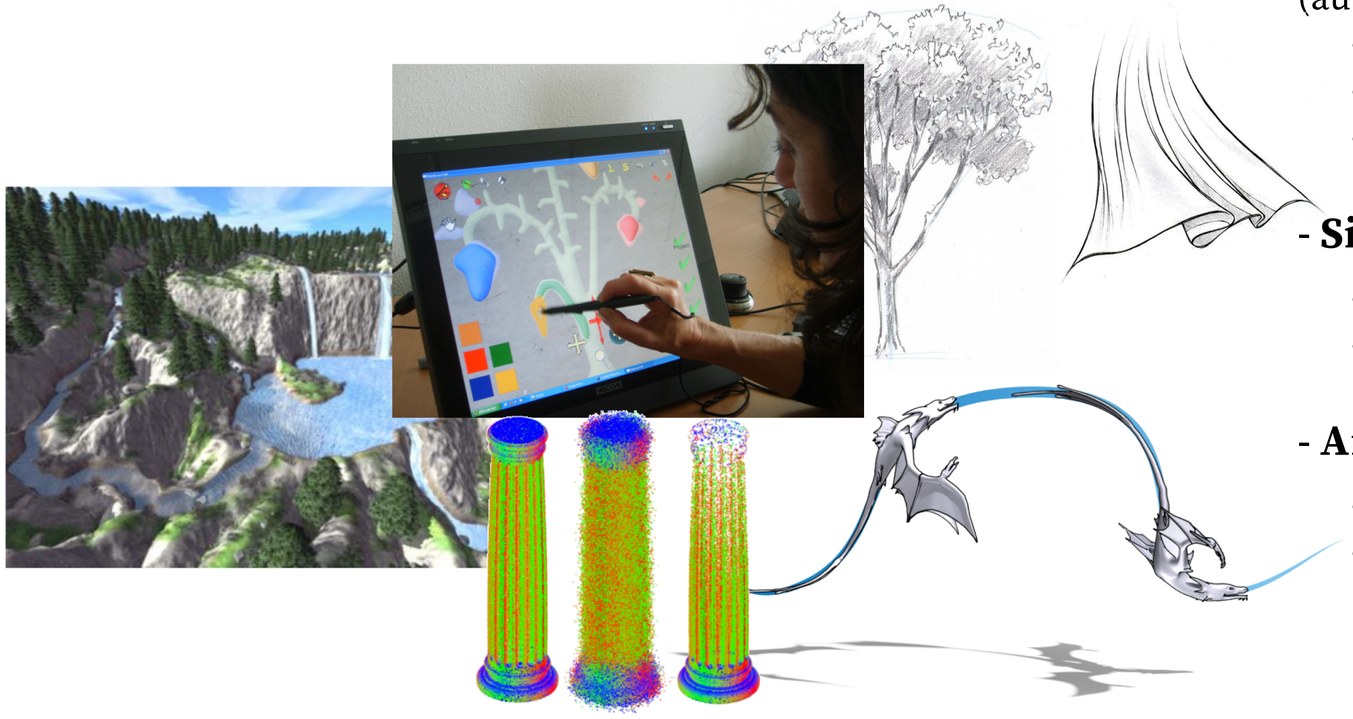


Domaine d'applications typiques

- **Loisirs & création artistique**
  (Cinéma d'Animation, VFX, Jeu Vidéo)

- **Modélisation & visualisation en Sciences Naturelles**

- **Prototypage et fabrication**



Notre "expertise"

- **Méthode interactive pour l'aide à la créativité**
(automatisation des tâches répétitives)
  - *Modélisation guidés par l'utilisateur: Esquisses, gestes, sculpture*
  - *Contraintes géométriques: Dévelopabilité, volume, etc.*
  - *Design et contrôle d'animation*

- **Simulation Visuelles**
  - *Couplage contraintes géométriques & simulation rapides*
  - *Animation de plis, froissement, détails, etc.*

- **Analyse de formes & algorithmique**
  - *Comparaison de formes et exploration*
  - *Apprentissage sur des données géométriques*

Notre groupe

Chercheurs permanents ($\times 4$)

- Marie-Paule Cani [Prof.]
- Pooran Memari [CR CNRS]
- Maks Ovsjanikov [Prof.]
- Damien Rohmer [Prof] [Resp. équipe]

Post docs ($\times 3$)

Ruqi Huang, Mikhail Panine, Amal Dev Parrakat.

Thésards ($\times 16$ in total, 12 on site)

Thomas Buffet, Renaud Chabrier, Nicolas Donati, Pierre Ecormier-Nocca, Maxime Kirgo, Maud Lastic, Mezghanni Mariem, Corentin Mercier, Pauline Olivier, Adrien Poulenard, Marie-Julie Rakotosaona, Abhishek Sharma.

Aide: stages, emploi, poursuite en Informatique Graphique?

Rem. IG: Domaine technique, R&D avancée

- Lien fort sujets recherche & entreprise.
- Thèses IG - sujets appliqués qui intéressent les industries.

Si le domaine vous intéresse:

AFIG - Association Française d'Informatique Graphique

https://www.asso-afig.fr/site/

Listing entreprises & *centre de recherches*.

Listing offres stages, thèse, ingénieurs.

+ N'hésitez pas à contacter les chercheurs.

Animation 3D - Plan du cours

1. **Animation descriptive [02/12/2019]**
   - Pipe-line de production 3D, animation expressive
   - KeyFraming
     - Interpolation de positions
     - Interpolation de rotations

   TP KeyFrame

2. **Animation physique [09/12/2019]**
   - Modèles (particules, solides, déformables)
   - Application simulation de particules
   - Simulation Tissus, simulation Fluides

   TP Simulation de tissus

3. **Animation de personnages [16/12/2019]**
   - Squelette d'animation: cinématique directe/inverse
   - Déformation de la peau: Skinning

   TP Skinning

Evaluation

A valider

- Un controle continu
  - un petit questionnaire papier $\simeq$ 15 min le 16/12/2019
  - sans documents ?
- Un compte rendu de TP
  - TP2 ou TP3 ?
  - $\simeq$ 5 pages

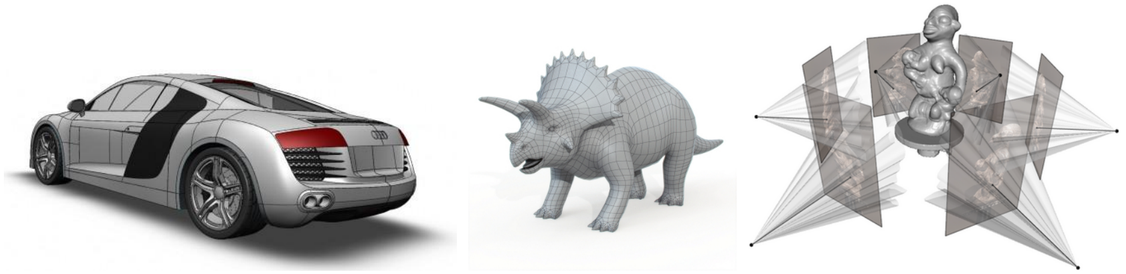
    Votre démarche, vos résultats, vos ***analyses***

Rappels d'Informatique Graphique

Computer Graphics main SubFields

Modeling

How to create static shapes

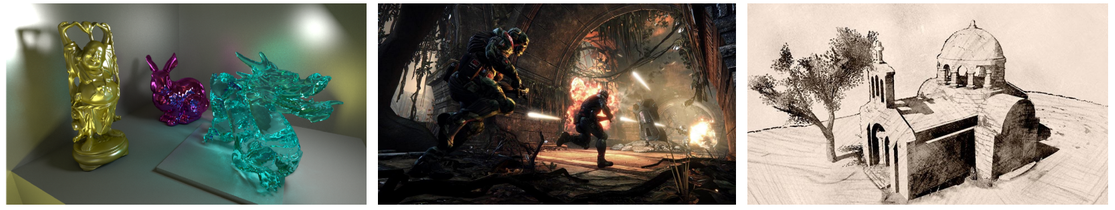


Animation

How to create and author time varying shapes

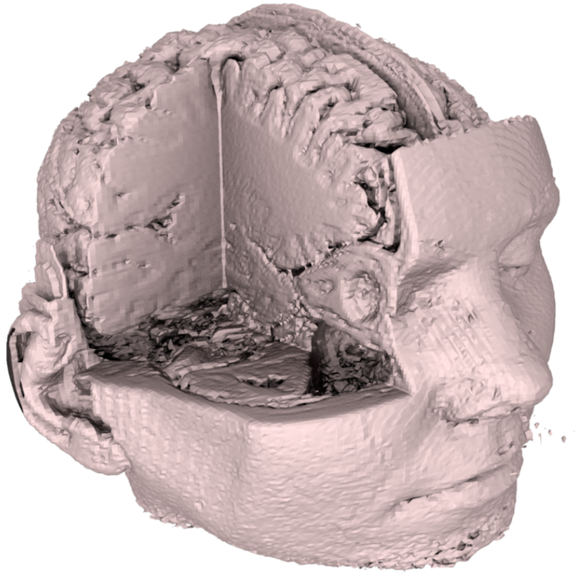
Rendering

How to generate 2D images from 3D data



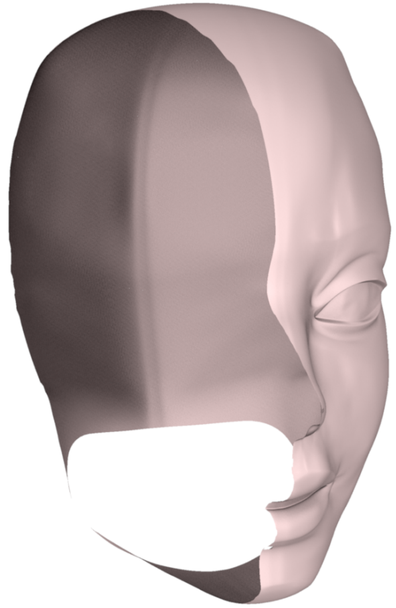
Representing 3D shapes for Graphics Applications

Volume representation



+ Accurate, handle density

Surface representation



+ Focus on visible part

+ Fast GPU rendering, low memory footprint

=> **Computer Graphics**: Mostly focus on representing **Surfaces**

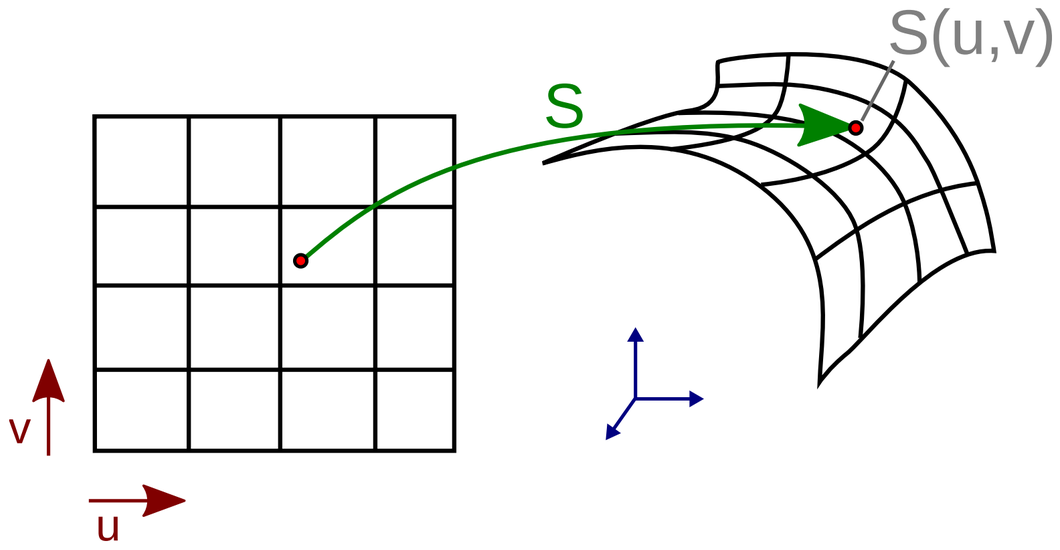
=> **Scientific visualization**: Volume data

Two main representations for surfaces

Explicit representation

$$S(u,v) = (x(u,v),\, y(u,v),\, z(u,v))$$

Parametric map



+ Neighborhood information

Implicit representation

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 0\}$$

Isosurface of scalar field

+ Topological modification

Two main representations for surfaces

Example for a sphere

Explicit representation

$$S(u,v) = (x(u,v),\, y(u,v),\, z(u,v))$$

Parametric map

$$S(u,v) = \begin{cases} x(u,v) = R\sin(u)\cos(v) \\ y(u,v) = R\sin(u)\sin(v) \\ z(u,v) = R\cos(u) \end{cases}$$

Implicit representation

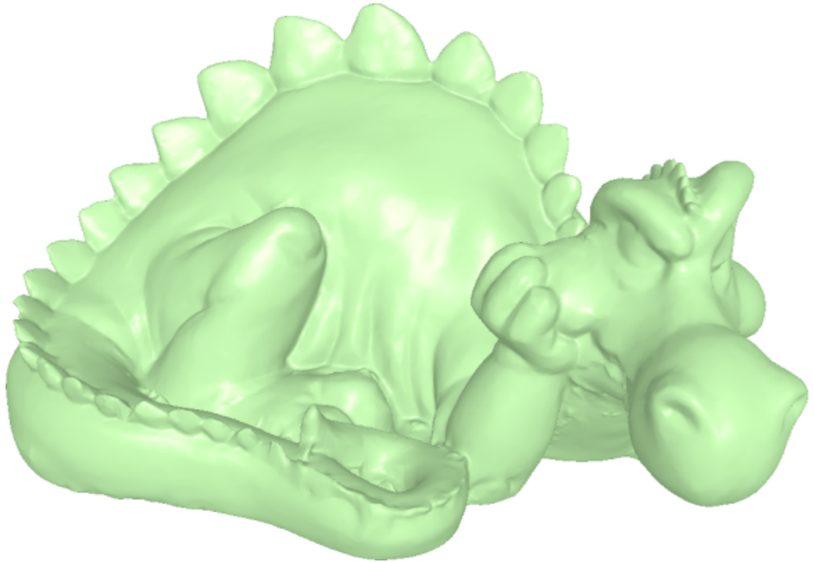
$$S = \{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 0\}$$

Isosurface of scalar field

$$F(x,y,z) = x^2 + y^2 + z^2 - R^2$$

Difficulty of surface representation using function

Which function can represent this shape ?



$$S(u, v) = ?$$

$$F(x, y, z) = ?$$

Objective of surface representation

Main Idea => Use of **piecewise approximation**

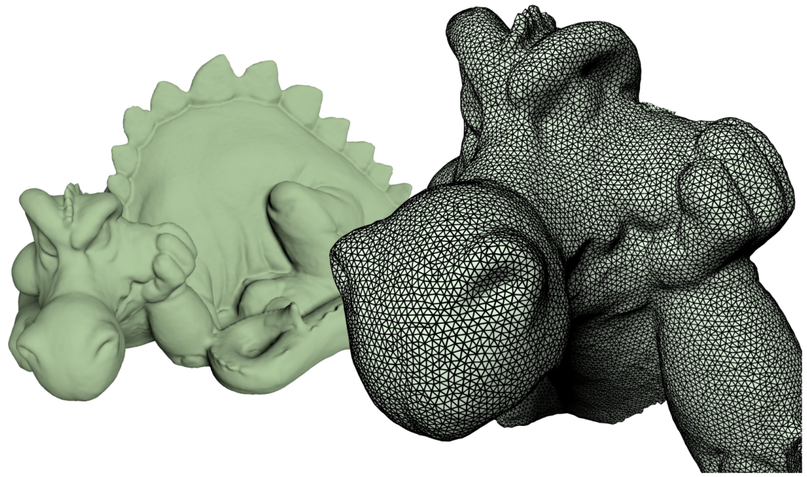
Ideal surface representation
- **Approximate** well any surface
- Require **few samples**
- Can be **rendered** efficiently (GPU)
- Can be manipulated for **modeling**

Example of models:
- *Mesh-based: Triangular meshes, Polygonal meshes, Subdivision surfaces*
- *Polynomial: Bezier, Spline, NURBS*
- *Implicit: Grid, Skeleton based, RBF, MLS*
- *Point sets*

=> For projective/rasterization render pipeline : always render **triangular meshes** at the end

+ Simplest representation
+ Fit to GPU Graphics render pipeline
- Requires large number of samples: complex modeling
- Tangential discontinuities at edges

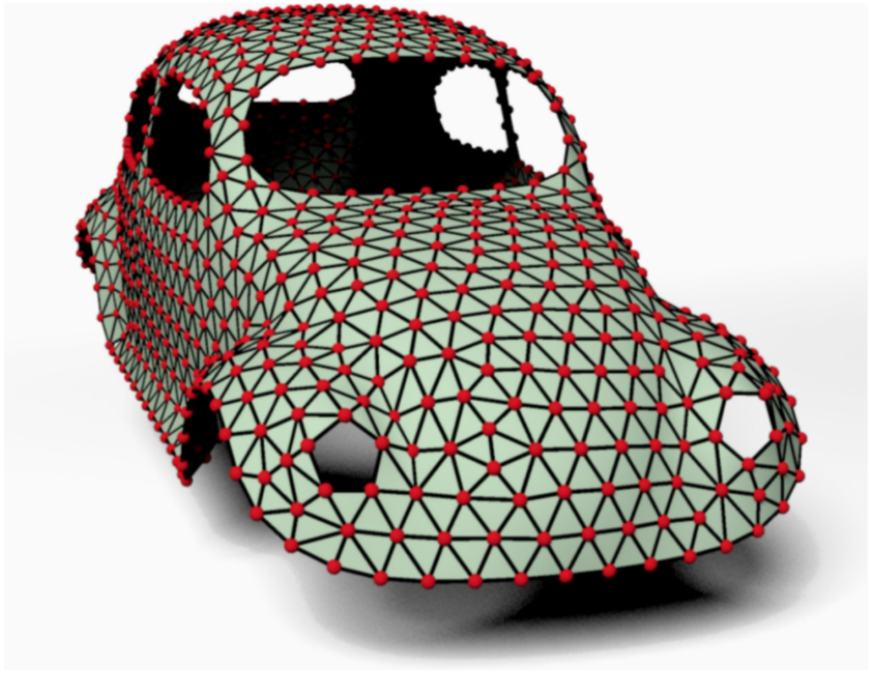
Meshes



Simplest possible representation of 3D surfaces: set of triangles

Described as a triplet: (Vertices, Edges, Faces)

$S = (\mathcal{V}, \mathcal{E}, \mathcal{F})$



● Vertex

$$\mathcal{V} = (v_1, \ldots, v_N)$$

／ Edge

$$\mathcal{V} = (v_1, \ldots, v_{N_e}) \in (\mathcal{V}^2)^{N_e}$$

◢ Face

$$\mathcal{F} = (f_1, \ldots, f_N) \in (\mathcal{V}^3)^{N_f}$$

Mesh encoding

Exemple for a tetrahedron

- 1st Solution: *Soup of polygons*

```
triangles = [(0.0, 0.0, 0.0), (1.0, 0.0, 0.0), (0.0, 0.0, 1.0),
             (0.0, 0.0, 0.0), (0.0, 0.0, 1.0), (0.0, 1.0, 0.0),
             (0.0, 0.0, 0.0), (0.0, 1.0, 0.0), (1.0, 0.0, 0.0),
             (1.0, 0.0, 0.0), (0.0, 1.0, 0.0), (0.0, 0.0, 1.0)]
```

- 2nd solution: *Geometry, Connectivity*

```
geometry = [(0.0, 0.0, 0.0), (1.0, 0.0, 0.0), (0.0, 1.0, 0.0), (0.0, 0.0, 1.0)]
connectivity = [(0,1,3), (0,3,2), (0,2,1), (1,2,3)]
```

=> Prefered solution

+ more space efficient

+ modifying 1 vertex = 1 operation

Mesh buffer encoding in C++

```cpp
#include <vector>
#include <array>

struct vec3 {float x,y,z;};

using index3 = std::array<unsigned int, 3>;

int main()
{
    std::vector<vec3> geometry = { {0.0f, 0.0f, 0.0f}, {1.0f, 0.0f, 0.0f},
              {0.0f, 1.0f, 0.0f}, {0.0f, 0.0f, 1.0f} };
    std::vector<index3> connectivity = { {0,1,3}, {0,3,2}, {0,2,1}, {1,2,3} };

    return 0;
}
```

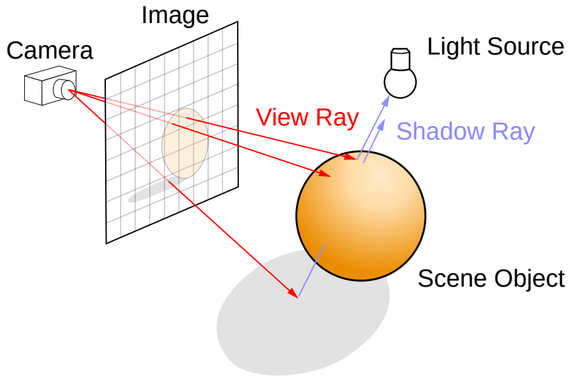
Example of 3D Mesh file

```
v -0.000000 0.620276 0.108446
v -0.000000 0.685780 0.104094
v 0.011128 0.685780 0.102245
v 0.014793 0.620276 0.106125
v 0.034724 0.684975 0.079817
v 0.040413 0.620278 0.086828
v 0.029160 0.619800 0.099405
v 0.024110 0.685194 0.093530
v 0.046714 0.554312 0.085764
v 0.033793 0.547222 0.100284
v 0.015067 0.542780 0.113608
v -0.000000 0.541146 0.117759
v 0.051177 0.430214 -0.047903
v 0.049948 0.435812 -0.035967
v 0.028863 0.449897 -0.050037
v 0.028839 0.444346 -0.059194
v 0.017691 0.251925 0.023686
v 0.034131 0.252216 0.014535
v 0.036689 0.275442 0.012672
v 0.015166 0.271140 0.025837
v 0.014869 0.285441 0.024957
```

Open question: How to display it efficiently on screen?

How to render surfaces

Ray tracing

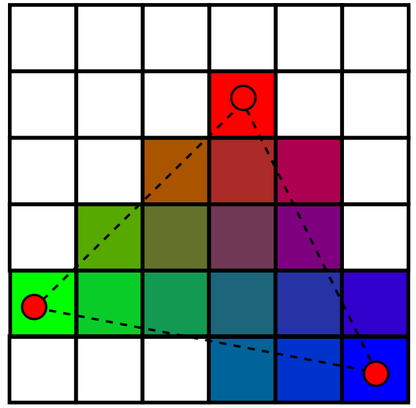


- *Throw rays from light-sources/camera*
- *Intersect rays with 3D shapes*
- *Pixel-wise computation*

+ Photo-realistic rendering
  (Soft shadows, reflection, caustics)
+ Handle general surfaces
- High computational cost

=> Restricted to offline rendering (but developing more and more)

Projection/Rasterization



- *Assume shapes made of triangles*
  1. *Project each triangle onto camera screen space*
  2. *Rasterize projected triangle into pixels*
- *Triangle-wise computation*

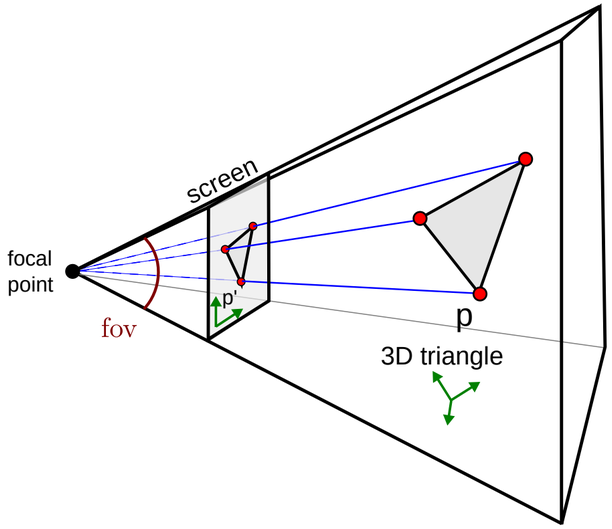
+ Efficiently implemented on GPU
- Limited to triangles
- No native effects (shadows, transparency, etc)

=> The standard real time rendering with GPU

Projection/Rasterization

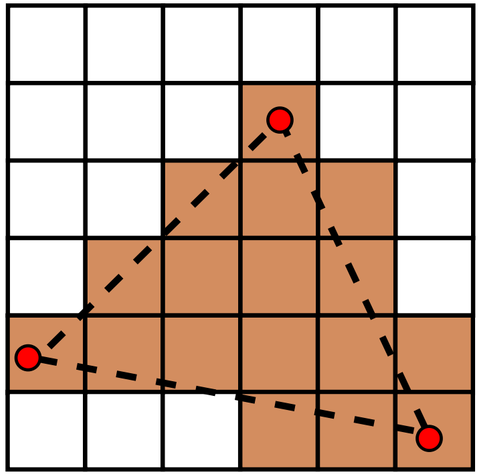
Object made of **triangles only**

1. **Project vertices of triangles**



- Projection computed as matrix operation (projective space $p' = \mathrm{M}\, p$)

2. **Rasterization**



- Discrete geometry
- Interpolate attributes (colors, etc) on each pixel

=> At interactive frame rate ($\geq$ 25 fps)

- Project all triangles of shapes
- Fill all pixels of each projected triangle

Quick fundamental notions for practical 3D programming

- Affine transform as 4D matrices
- Perspective and projective space
- Illumination and normals

Affine transforms and 4D vectors/matrices

Preliminary note

- We use a lot affine transformations to place shapes in 3D space

  Translation, Rotation, Scaling

- In CG vectors are often expressed in 4D, and matrices are 4×4.

=> Reason: Affine transforms can be expressed linearly (with matrices) in 4D

$$p = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \qquad \mathrm{M} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & t_x \\ m_{10} & m_{11} & m_{12} & t_y \\ m_{20} & m_{21} & m_{22} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Affine transform in 2D

General principle in the 2D case

Example for a point $p = (x, y)$

$$\text{Rotation } \mathrm{R} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \text{Scaling } \mathrm{S} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \text{Translation } (x + t_x, y + t_y) \text{ } \textit{(not linear)}$$

Cannot express conveniently composition b/w several rotation, scaling, translation.

Trick - Add an extra coordinates to points $p = (x, y, 1)$ (homogeneous coordinates).

$$\text{Then translation can be expressed linearly } p' = \mathrm{T}\, p, \text{ with } p' = \underbrace{\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}}_{\mathrm{T}} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

$$\text{Similarily with rotation } \mathrm{R} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and scaling } \mathrm{S} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Affine transform matrix

With the extra dimension (in 2D):

Translation T, rotation R, scaling S can be composed as matrix products representation

ex. $\mathrm{M} = T_0\, R_0\, S_0\, T_1\, R_1\, S_1\, \ldots$ $\quad \mathrm{M} = \left(\begin{array}{cc|c} m_{00} & m_{01} & t_x \\ m_{10} & m_{11} & t_y \\ \hline 0 & 0 & 1 \end{array}\right)$.

m_{ij} : linear part (rotation and scaling); $t_{x/y}$: translation part

Similar in **3D** but with **4-components vectors**, and $\mathbf{4 \times 4}$ **matrices**.

$p = (x, y, z, 1)$ - *represents 3D position*

$\mathrm{M} = \left(\begin{array}{ccc|c} m_{00} & m_{01} & m_{02} & t_x \\ m_{10} & m_{11} & m_{12} & t_y \\ m_{20} & m_{21} & m_{22} & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$ - *represents 3D affine transformation (rotation, scaling, translation)*

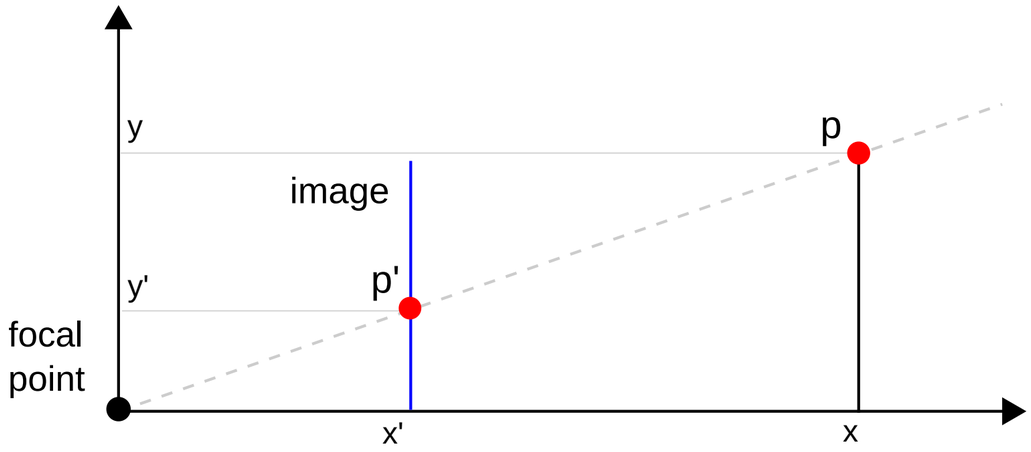
Note: vectors and points can be expressed

- 3D point $(x, y, z, 1)$ - translation applies.
- 3D vector $(x, y, z, 0)$ - translation doesn't apply.

Perspective and projective space

Modeling perspective projection requires division.

ex. in 2D (1D projection)



$$y' = x' \frac{y}{x} = f \frac{y}{x} \quad (f\text{: focal})$$

Linear model using 3D vectors in projective space.

$$p' = \begin{pmatrix} f \\ f\frac{y}{x} \\ 1 \end{pmatrix} \underbrace{=}_{normalization} \begin{pmatrix} fx \\ fy \\ x \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

considering that the last coordinate must always be normalized to 1 (for points).

Projective space

- *Real* points lie on $z = 1$
- Vectors lie on $z = 0$

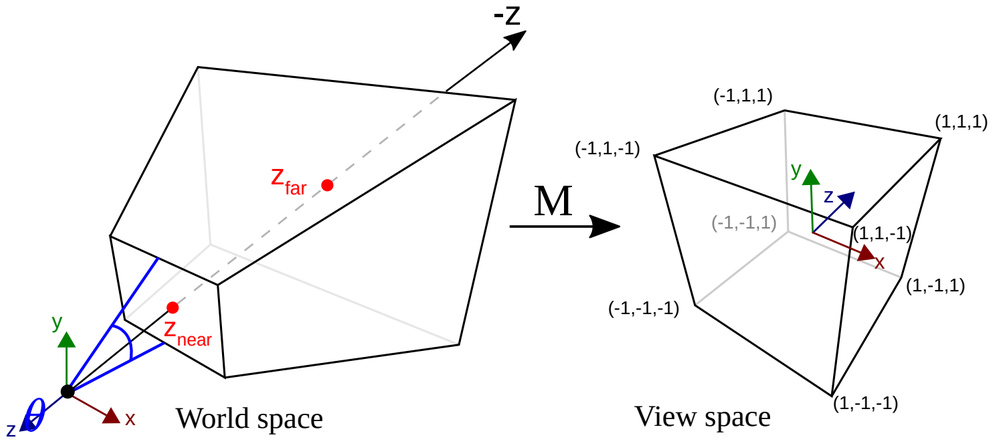
Real coordinates of points are obtained after normalization (division by z).

Perspective matrix

Perspective space : Allows perspective projection expressed as matrix.

Common constraints (in OpenGL)

- Wrap the viewing volume (truncated cone with rectangular basis called *frustum*) $(z_{near}, z_{far}, \theta)$ to a cube.
  - θ: view angle
  - $p = (x, y, z, 1) \in$ frustum $\Rightarrow p' = (x', y', z', 1) \in [-1, 1]^3$.



$$\mathrm{M} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$f = 1/\tan(\theta/2)$$
$$L = z_{near} - z_{far}$$
$$C = (z_{far} + z_{near})/L$$
$$D = 2\, z_{far}\, z_{near}/L$$

In practice

=> You must define z_{near}, z_{far}

=> $z_{far} - z_{near}$ should be as small as possible for maximum depth precision.

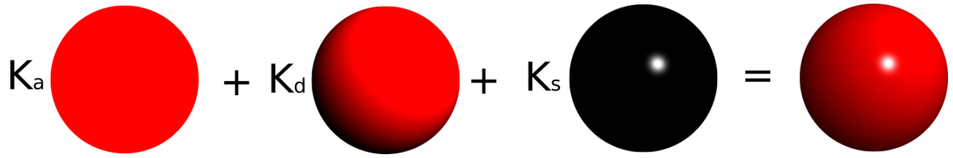
To which view space coordinates are mapped 3D world space points at z_{near}, z_{far} ?

Per-vertex normal and illumination

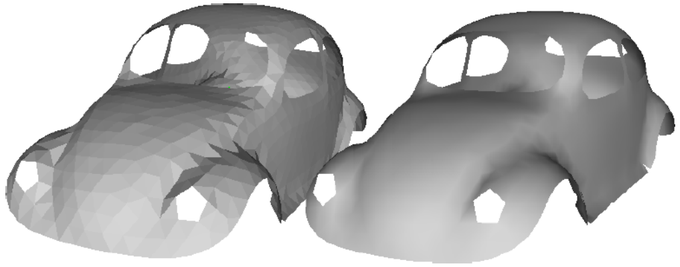
For smooth looking meshes, we define a **normal per-vertex**.

- Vertices are seen as samples on a smooth underlying surface

- Normals are used for illumination
  ambiant, diffuse, specular components

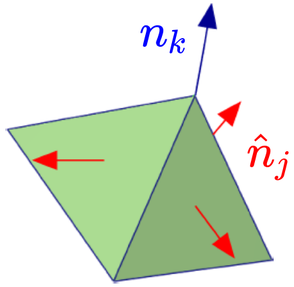


- *Phong shading* interpolates normals on each *fragments* of triangles, and compute illumination.



Possible automatic computation of normals: averaged normals of surrounding triangle.

$$n_k = \frac{\sum_{j \in \mathcal{V}_k} \hat{n}_j}{\| \sum_{j \in \mathcal{V}_k} \hat{n}_j \|}, \mathcal{V}_k \text{: neighboring triangles.}$$

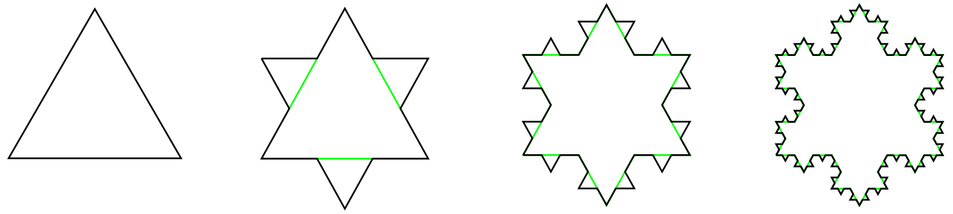


Fractals

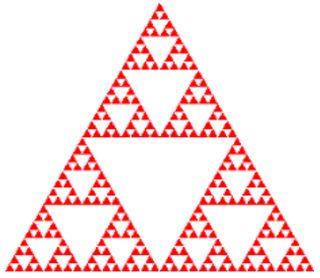
Idea: Recursively add self-similar details

Simple rule ⇒ complex shapes

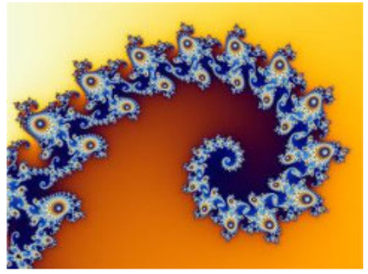
May look like complex natural details



Koch Snowflake



Sierpinski triangle, Shell:Oliva Porphyria



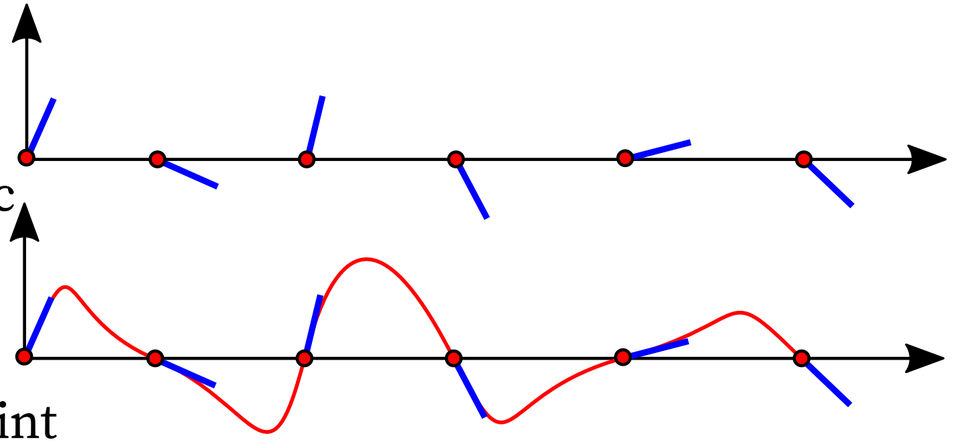
Perlin Noise

A widely used *noise* function.

Original article [An Image Synthesizer, Ken Perlin, SIGGRAPH 85]

Continuous Pseudo-random function

- Spatial variations are continuous, but non periodic
- Function can be evaluated at any point - deterministic

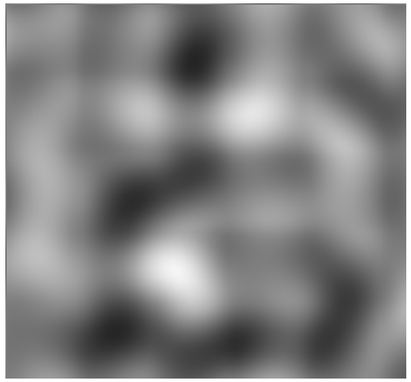


Creating a smooth function

- Compute pseudo-random gradient at each sample point

  Use some hash function

- Interpolate smooth cubic curve between each points



(Algorithm for nD: Simplex noise)

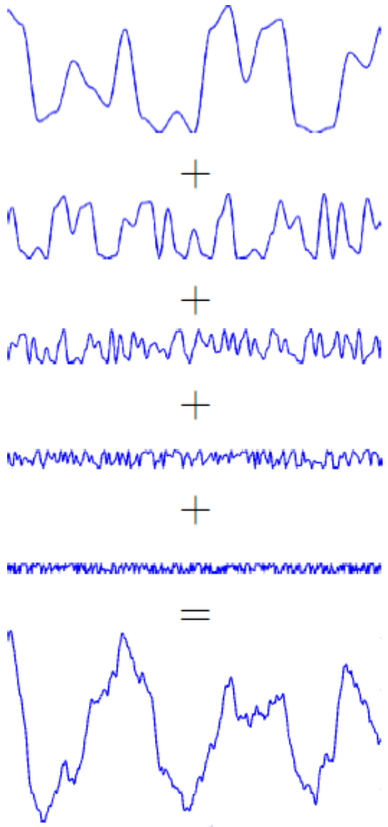
Fractal Perlin Noise

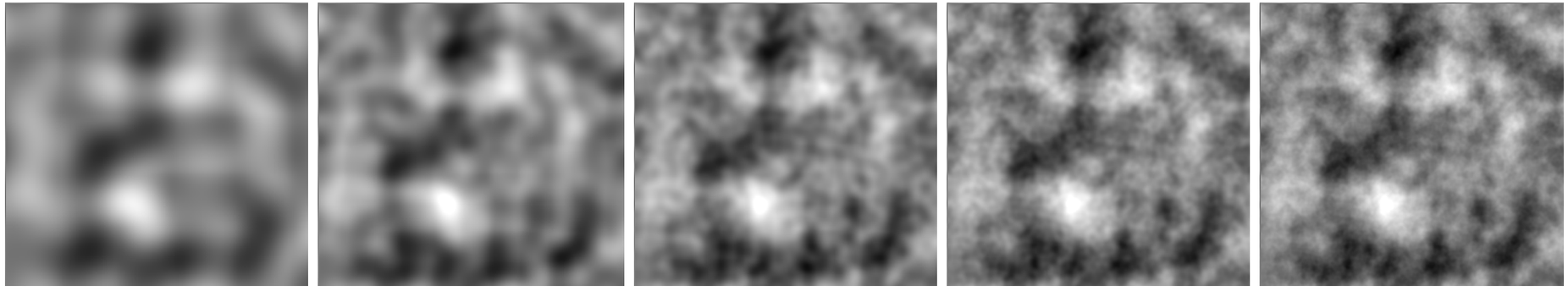
Sum over mutliple instance with increasing frequencies

f : Smooth Perlin Noise function

$$g(x) = \sum_{k=0}^{N} \alpha^k f(\omega^k \ x)$$

- N number of Octave
- α attenuation ($1/\alpha$ persistence)
- ω frequency gain

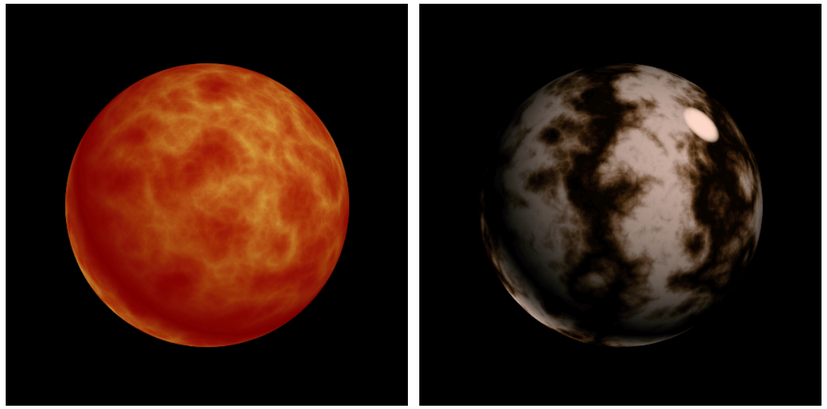




Perlin noise usage

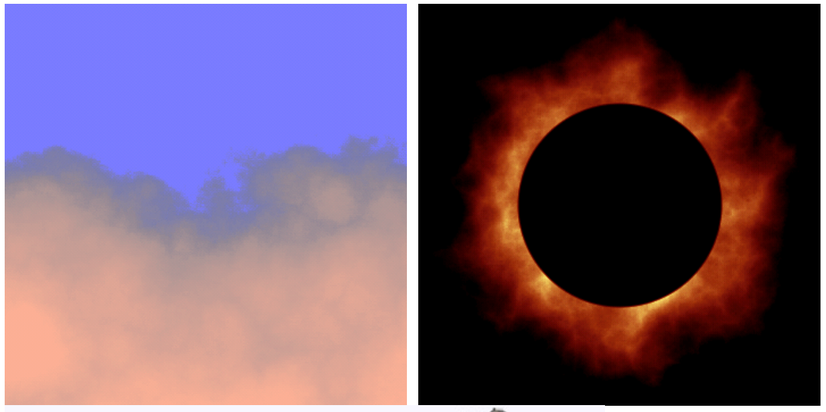
Material texture

- Ridge effect: $\sum_k \alpha^k |f(\omega^k x)|$
- Marble effect: $\sin\left(x + \sum_k \alpha^k |f(\omega^k x)|\right)$



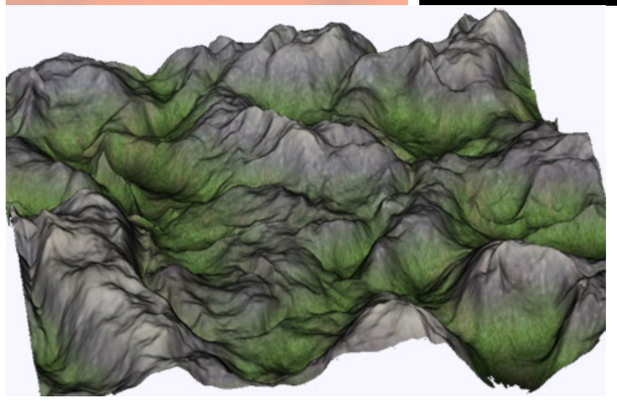
Animated textures

- Translation: $f(x, y + t)$
- Smooth evolution: $f(x, y, t)$



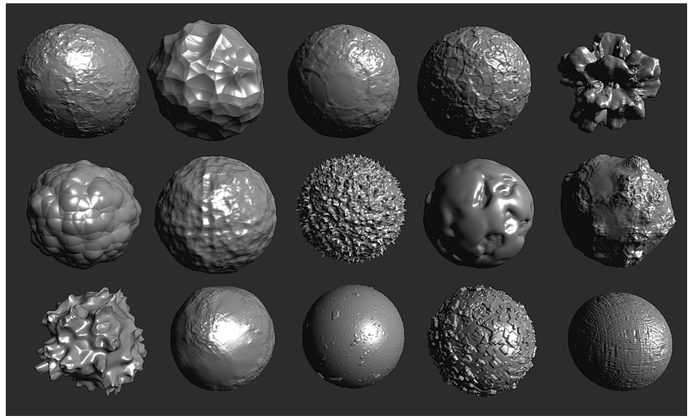
Mountain-looking terrain

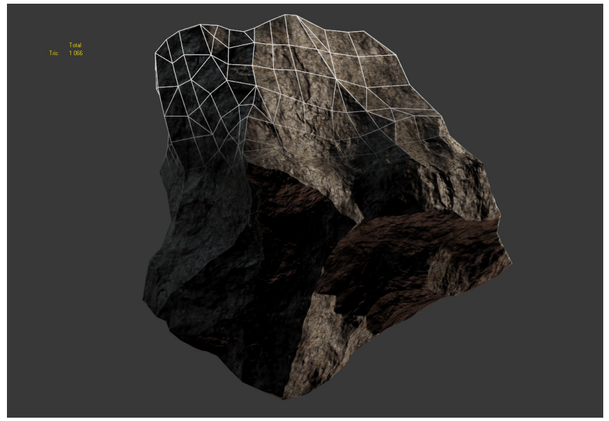
- $z = f(x, y)$

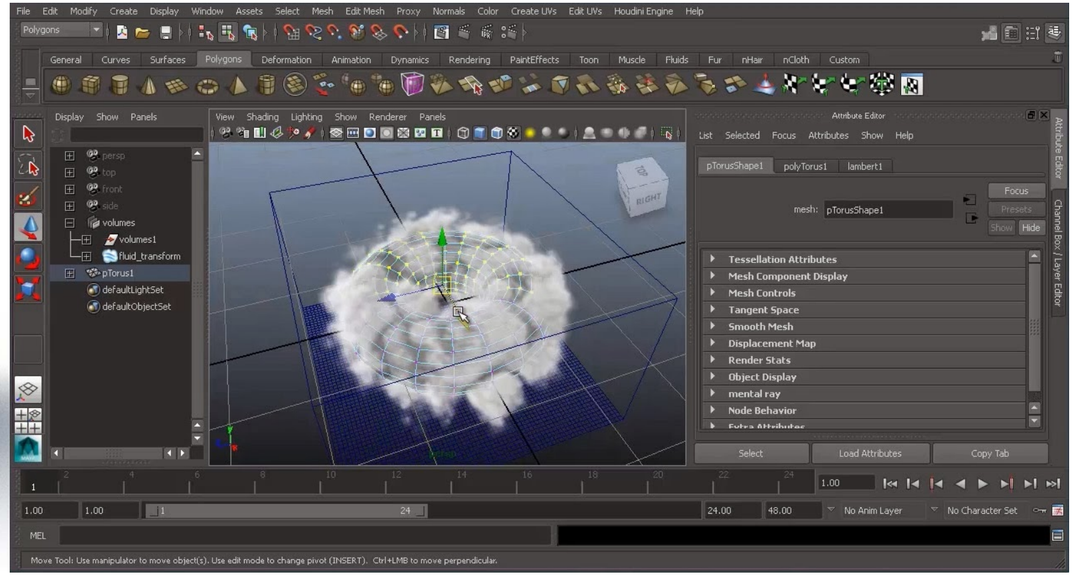


Perlin noise applications

In almost every complex shapes ...







Look at Shader Toy + Noise (example)

Perlin Noise Terrain

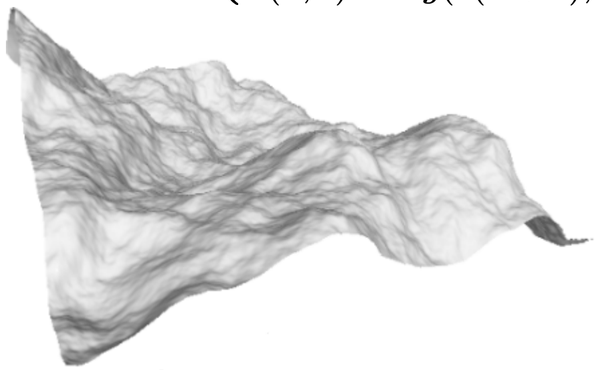
Consider the surface S defined as

$$S(u,v) = \begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = h\, g(s(u+o), s(v+o)) \end{cases}$$

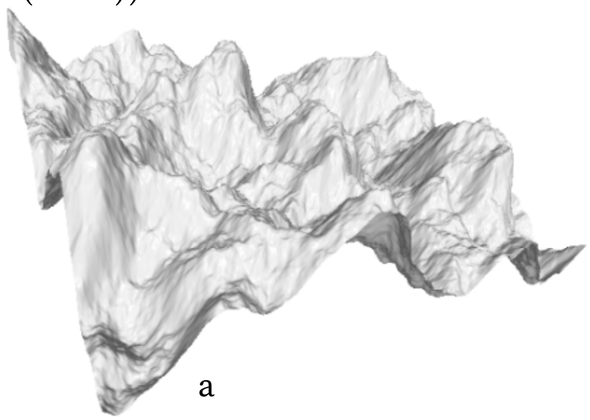
The perlin noise

$$g(u,v) = \sum_{k=0}^{N} \alpha^k \, f(2^k \, u, 2^k v)$$

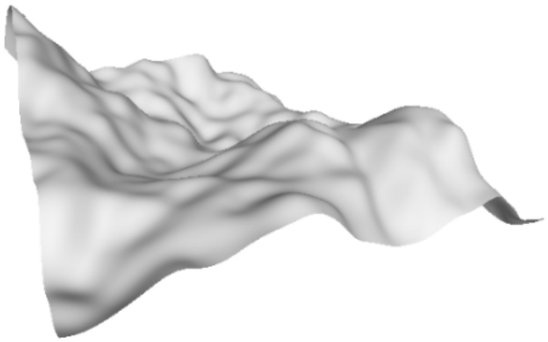
$N = 9$
$\alpha = 0.4$
$h = 0.3$
$s = 1$
$o = 0$



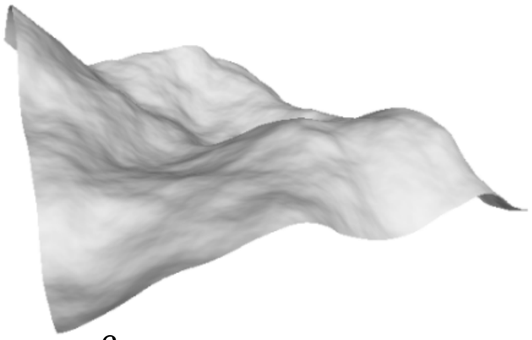
reference



a

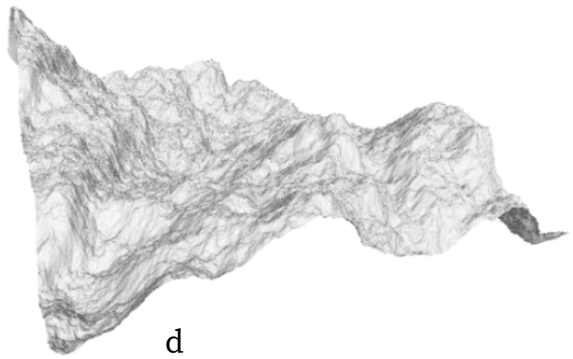


b

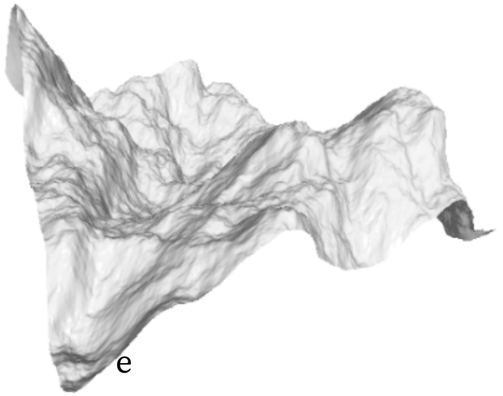


c

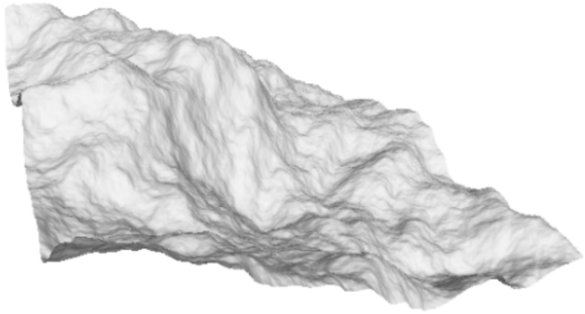
Which parameters correspond to (a,b,c,d,e,f) terrains?



d



e



f

Introduction to Computer Animation

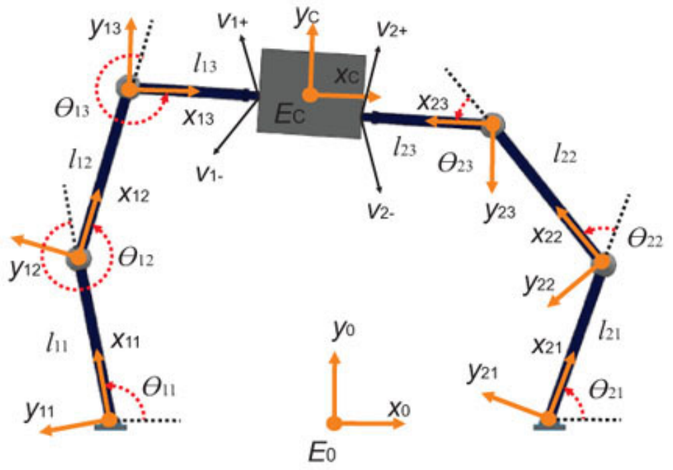
Animation in production

Animation in Computer Graphics

Two main ways to describe animation

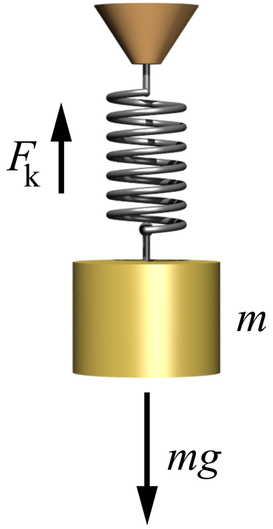
1- Kinematics

Shape deformation/motion without knowledge of the physical cause of the motion



2- Dynamics

Motion through forces acting on shapes (ex. physically based simulation)



Animation in Computer Graphics

Four ways to generate animation

1- Descriptive animation

2- Acquired motion

3- Procedural generation

4- Physically based simulation

Descriptive Animation

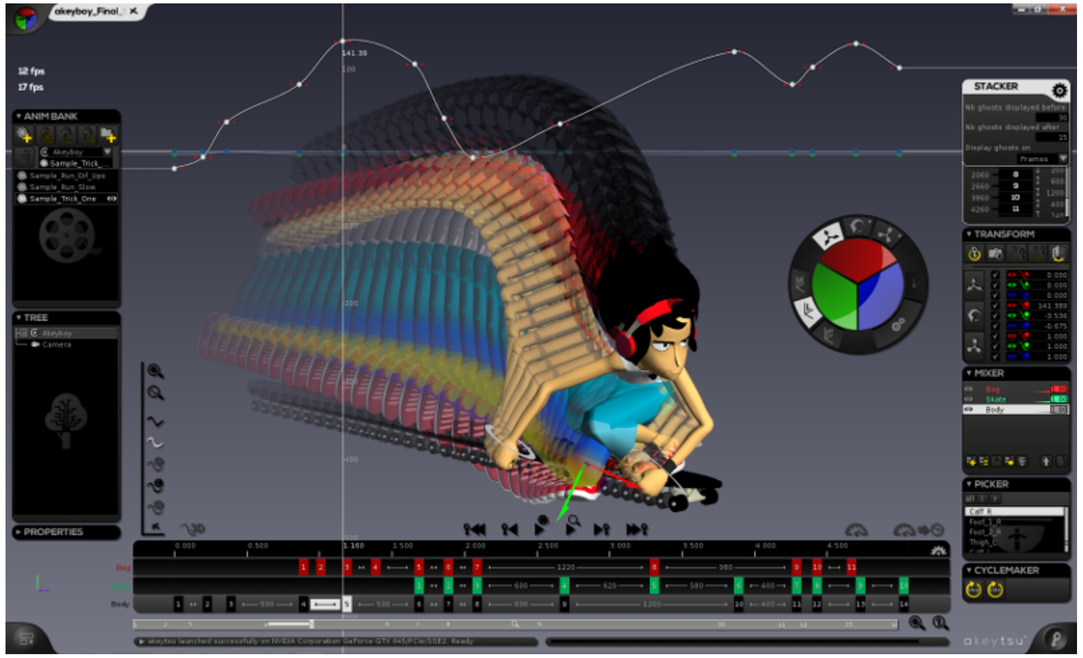
The artist/an algorithm fully describes the motion and deformation.

pro

+ Full control on the result

cons

- May introduce un-physical artifacts



History of CG Animation

First production of animated films (Disney)

- 30 drawings / sec

Principle

- *Animator in chief*: Create **Key Frames**
- *Assistants*: Fill the *in between* (secondary drawings)

=> Principle of **Key Framing**

(c) Walt Disney Company, from "The Illusion of Life"

Key Framing in CG

- Create *manually* a set of **Key Frames** at specific time steps
- Compute automatically intermediate frames (30 fps) using **interpolation**

Animator Terminology

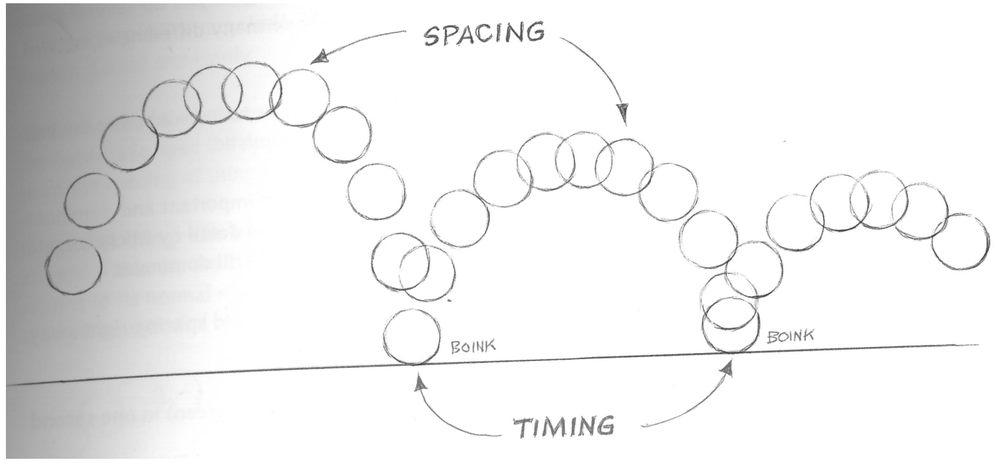
1- **Posing** the *key frames*

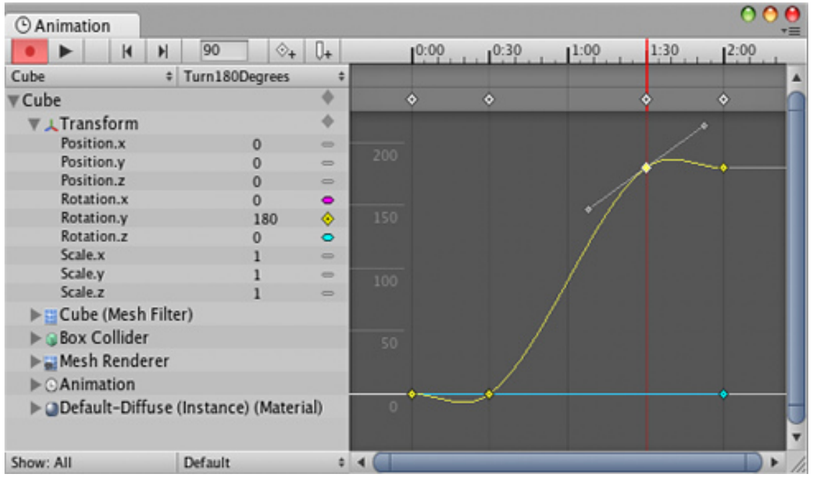
Set the main general posture of the character

*Linked to geometric **character deformation**, time is not involved*

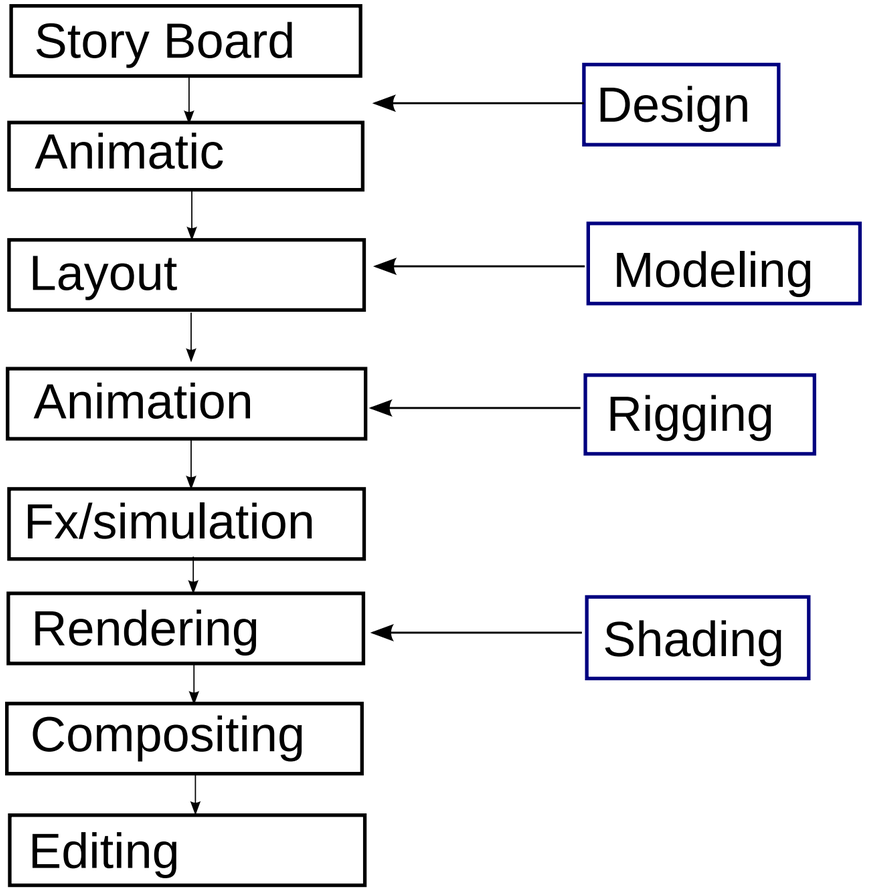
2- **Animating** the in-betweens

- **Timing :** Place key frames at specific times (global length of an action)
- **Spacing :** Speed of the interpolation (dynamic of the action)





Animation in production



- Animators: about 3/4 of production artists team.

- One professional animator can create 1-10s animation per day.

"MONSTERS UNIVERSITY" Progression Image 1 of 6: STORY
©2013 Disney•Pixar. All Rights Reserved.

"MONSTERS UNIVERSITY" Progression Image 4 of 6: LAYOUT
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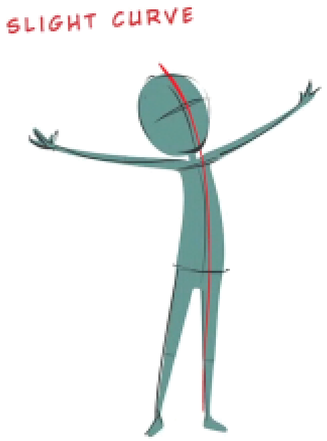
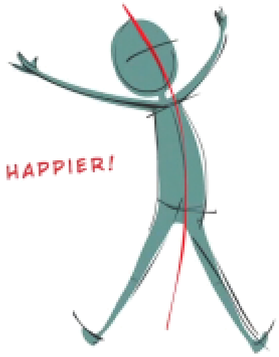
Animation in production - example

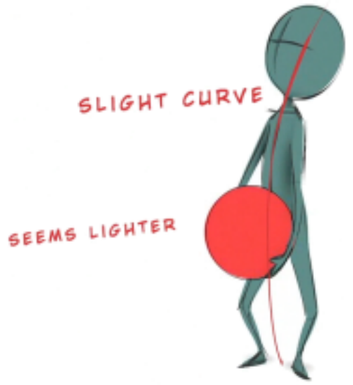
- In production *Animation* refers only to rigged character animation
- All other effects (dynamic hair, cloth, explosions, etc) are called *special effects*.

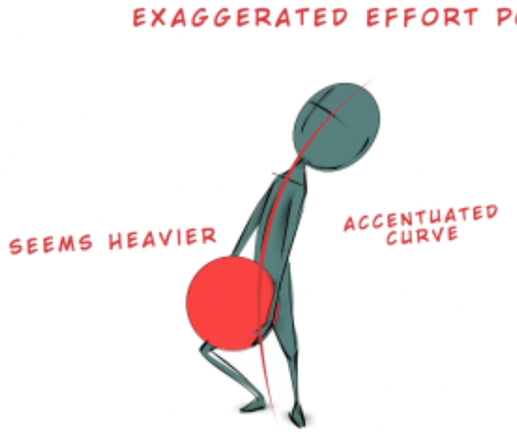
Character Animation Posing - Line of Action

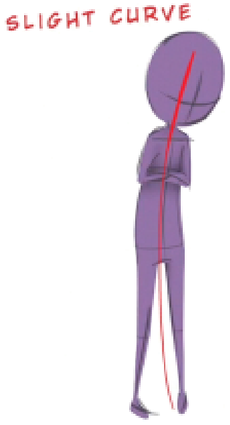
- Line of action: *Medial axis* expressing the character pose
- Express *statically* the dynamic of the action
  - Unstable pose ⇒ Dynamic action/motion

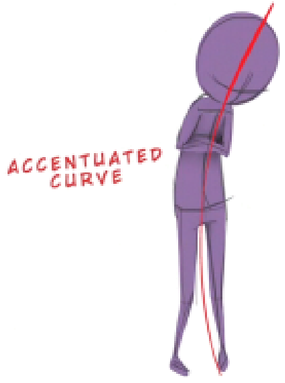












Expressive animation

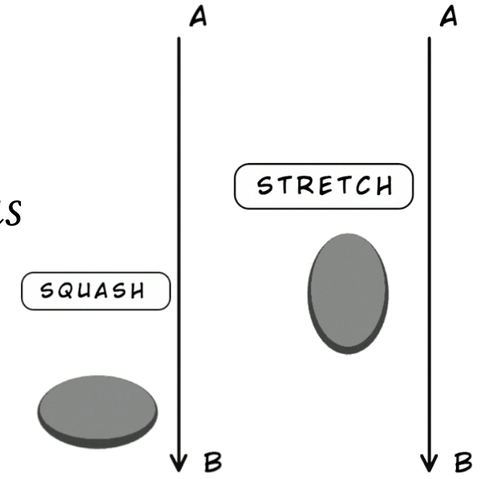
- Interpolation between realistic poses isn't enough for expressive animation
- *12 principles of animation* by Disney *Illustion of Life, 1981*

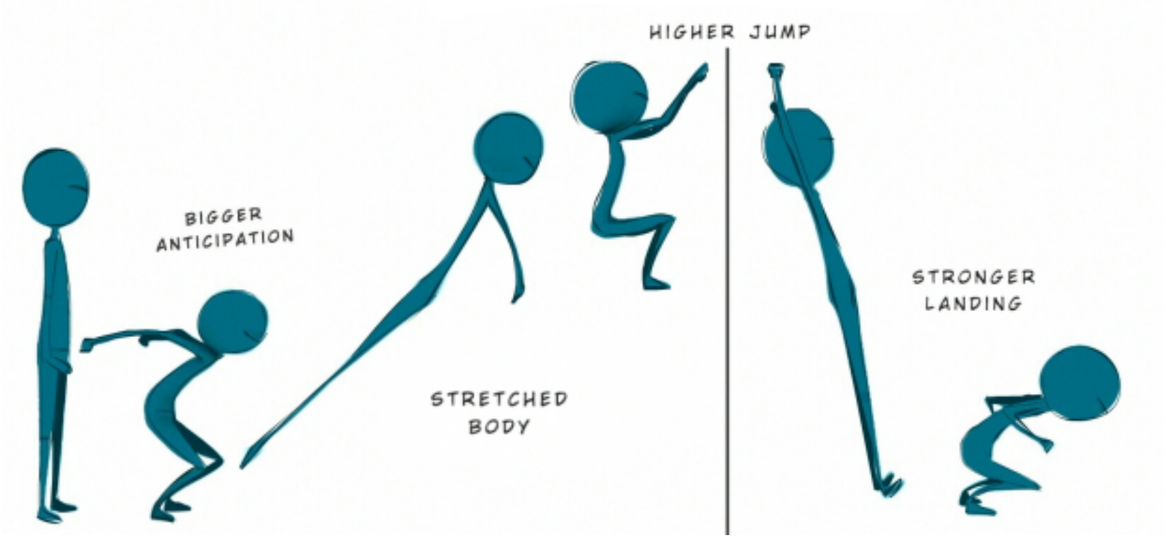
1. Timing
2. Spacing
3. Slow-in, Slow-out
4. Squash & Stretch
5. Anticipation
6. Follow Through
7. Secondary Action
8. Exaggeration
9. Appeal
10. Arcs
11. Staging
12. Straigh Ahead/Pose to Pose

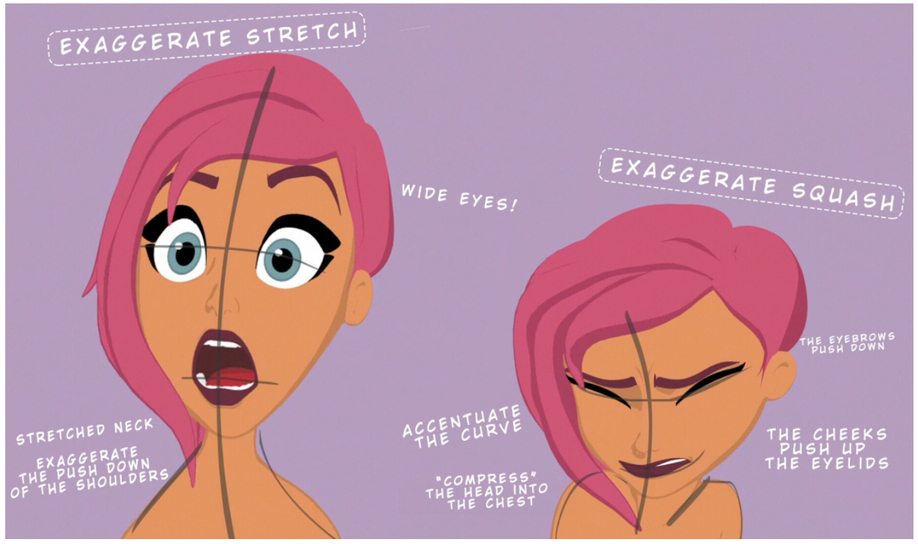
Expressive animation

Squash & Stretch

- Very common in cartoon
- Unrealistic, but surprizingly *plaus*



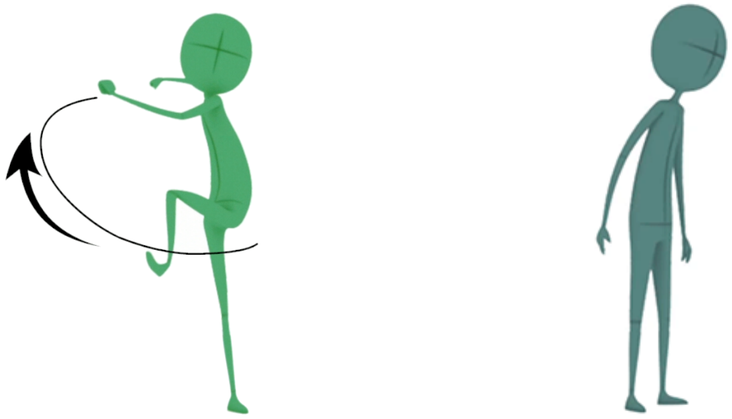


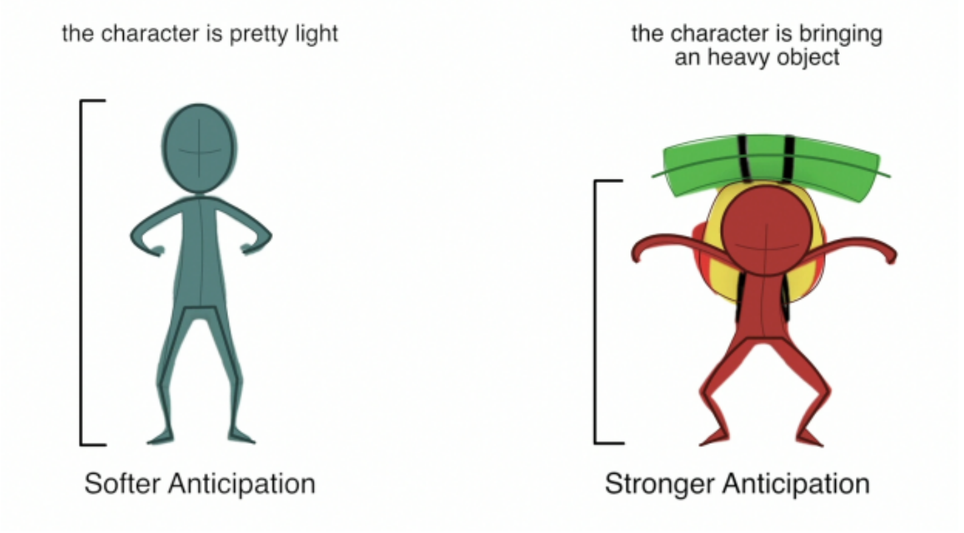


(C) Chiara Porri

Expressive animation

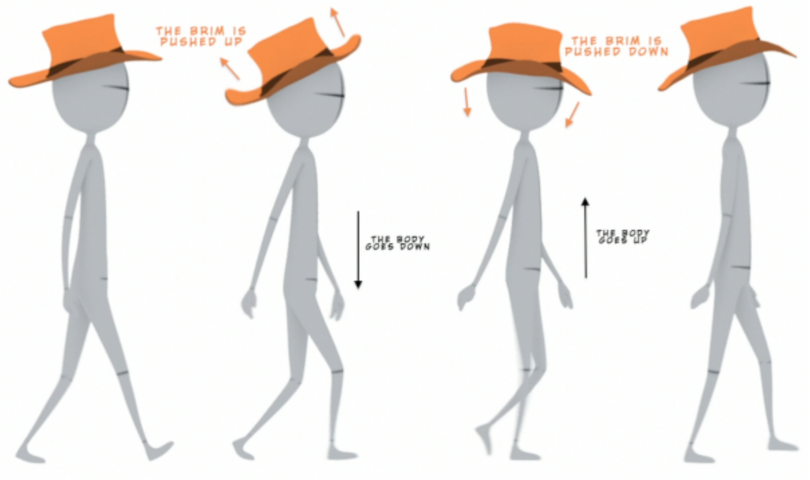
Anticipation

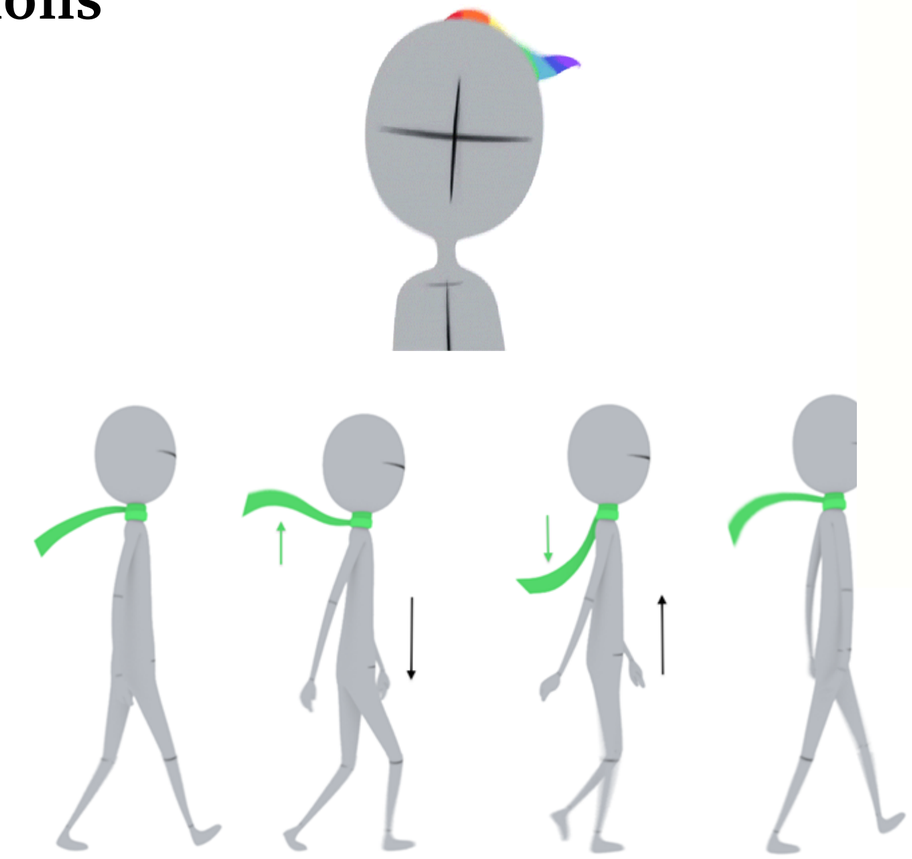


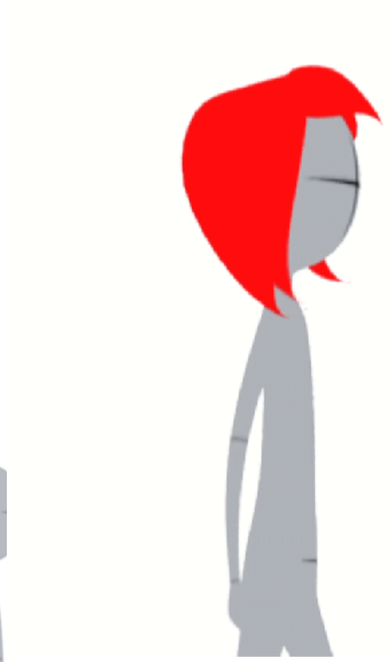


Expressive animation

Follow Through / Secondary motions







(C) Chiara Porri

Particle system

Particle system

Particle : element at a given position + extra parameters (mass, life time, etc)

Called **particles systems** in opposition to

- Rigid bodies - Solid objects with static shape
- Deformable bodies - Continuum material that can deforms

Pro

(+) Lightweight representation (both CPU+memory)

(+) Generic flexible model (spatial deformation, no connectivity, etc)

$\Rightarrow$ highly used in CG

Cons

(-) Simple model from physics point of view

Particles systems History

One of the first animated model in CG

Star Trek II

[Particle systems - A Technique for Modeling a Class of Fuzzy Objects, William T. Reeves, Lucasfilm, 1983] [pdf]

Karl Sims, Particle dreams, 1988 [link]

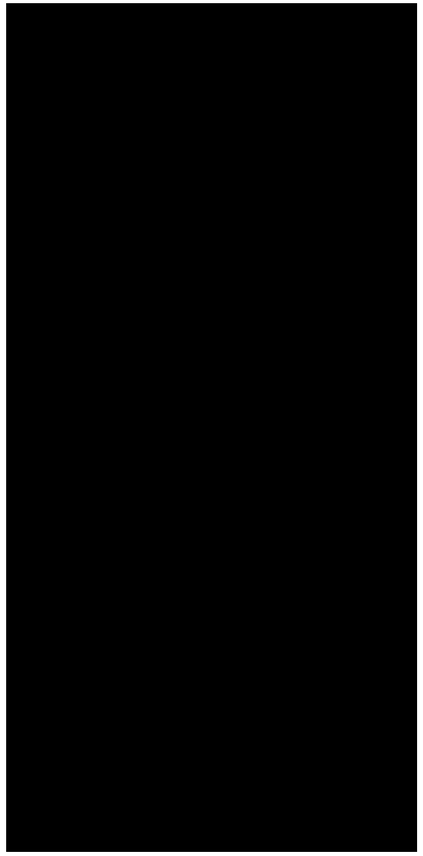
Example of Particle system

Free fall of spheres under gravity

- Geometrical representation of each particle: sphere
- Equation of motion $p(t) = \frac{1}{2}gt^2 + v_0 t + p_0$

- Initial position and speed may be placed at random positior
- Each particle may have a different life time

- *What are the parameters used for p_0 and v_0 in this example ?*



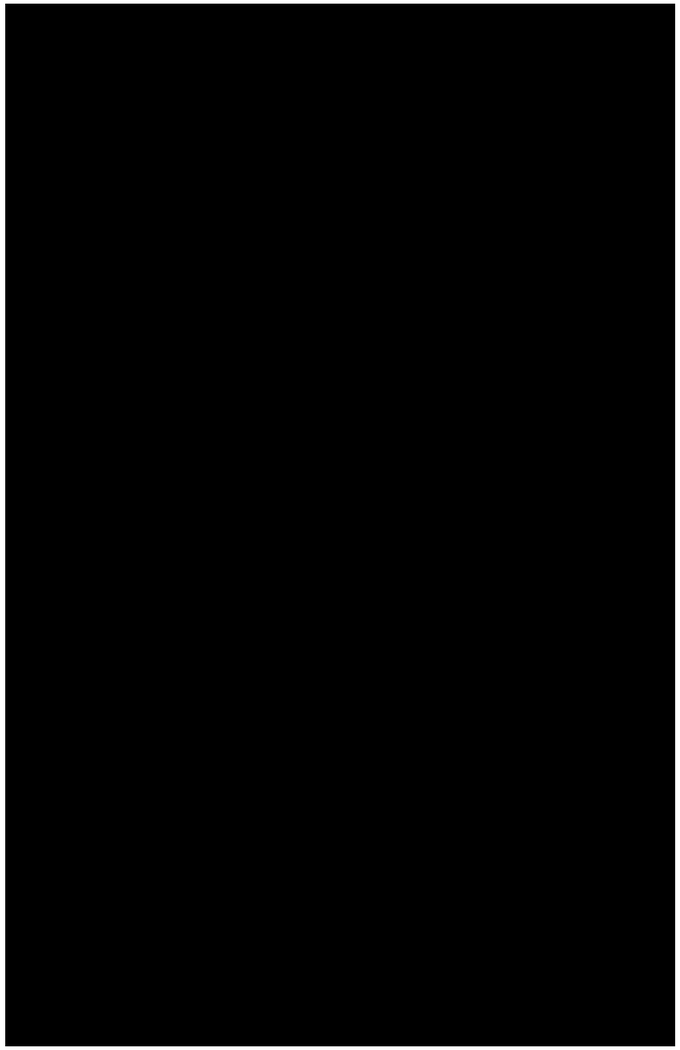
Bouncing spheres

- What is the equation of motion (taking into account the bouncing) ?

  - *Consider a particle emited at time* $t = 0$
  - *At what time* t_i*, the particle touch the floor ?*
  - *What is the new speed after impact ?*

General motions

Motion equation is not restricted to physics-based equations



- *What are the corresponding equations of motion ?*

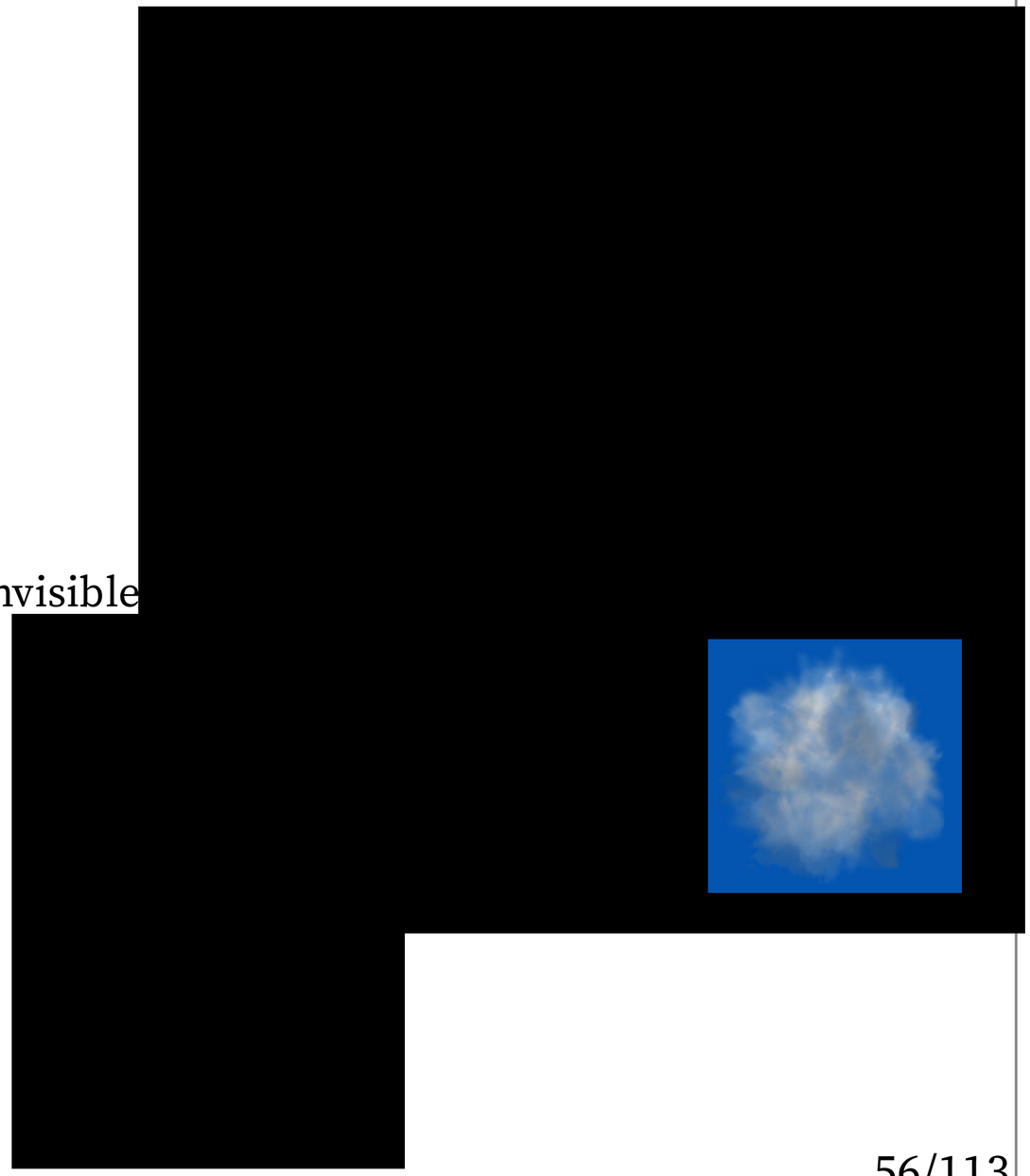
Billboards, Impostors, Sprites

Particles can be displayed as small images/thumbnails

In practice

- Each particle is displayed as a quadrangle
- A texture is mapped on the quad

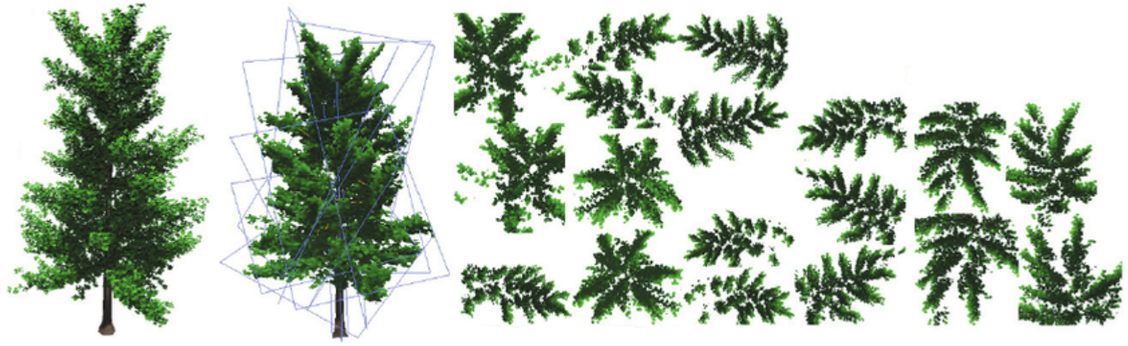
- The texture can contains transparency : quad geometry is invisible
- The quad can be facing the camera at all time (billboard)
- The texture can be animated (sprite)
- Texture can be adapted to the point of view (impostors)

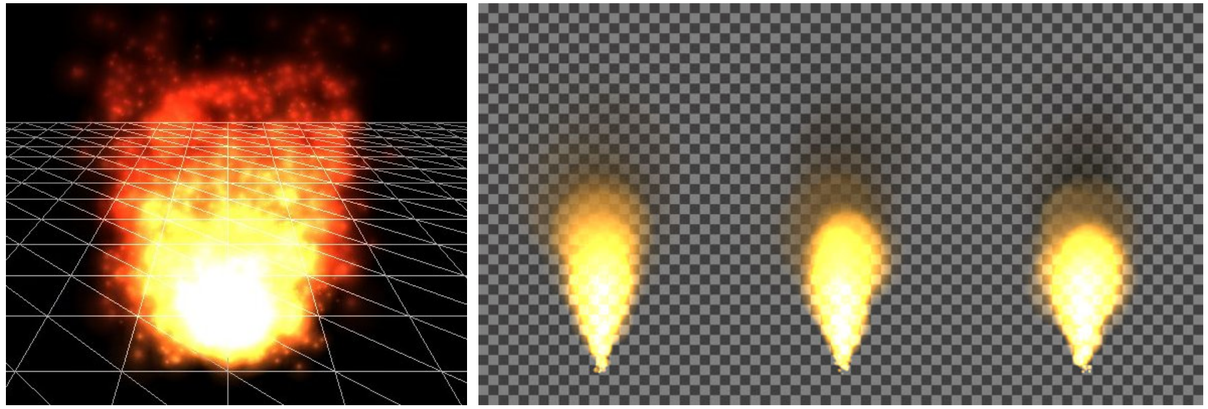


Usage of billboards

Large use of billboards for complex models

Vegetation, fire, smoke, etc.





Use case in production

How to model the *horse heads made of water* ?

The Lord of the Rings

Use case in production

How to model the *horse heads made of water* ?

The Lord of the Rings

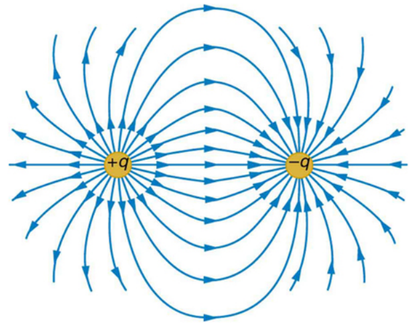
Use case in production

Full Making-Of

The Lord of the Rings

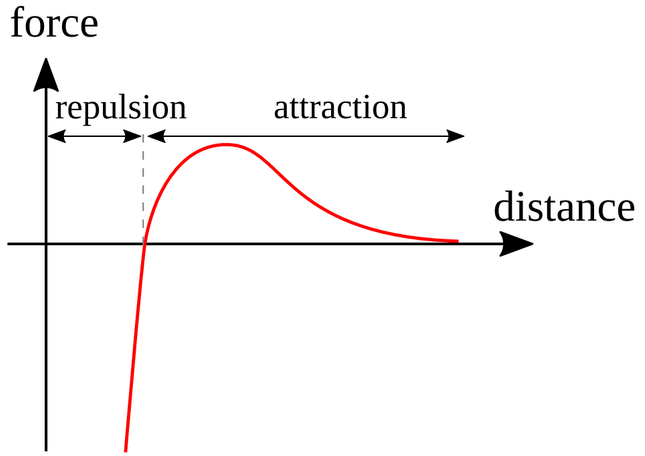
Interaction between particles

Interaction as **force field** (interact at distance)

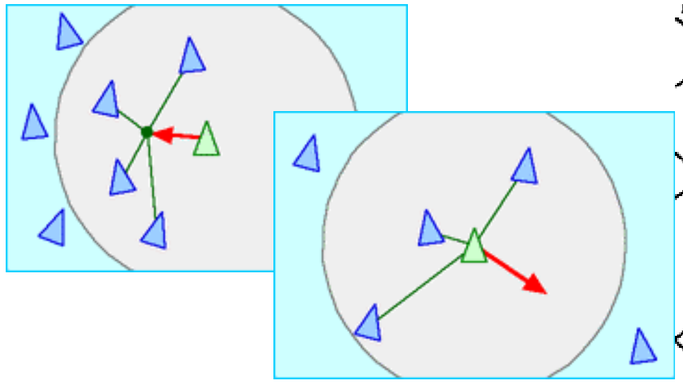


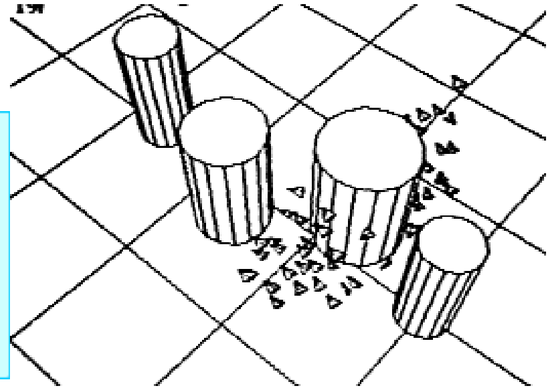
Example of usage

- Models crowd of life-like characters at large scale
  Inspired from physics particles forces (ex. Lennard-Jones potential)
  Attraction at *long-range*
  Repulsion at *short-range*



- First model: **Boids** Craig Reynolds 1987
- Extended later to human crowd modeling





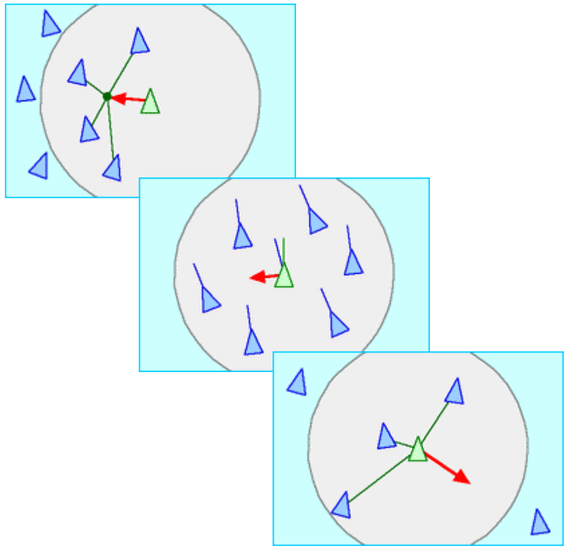
Boids Model

Introduced by
- [*Craig Reynolds. Flocks, Herds, and Schools: A Distributed Behavioral Model, SIGGRAPH 1987*] [link]
- [*Craig Reynolds. Steering Behaviors For Autonomous Characters. Proceedings of Game Developers, 1999*] [link]

A boid is defined by its
- Position
- Speed
- Forces acting on it

Three basic local *steering behaviors* to model
- **Cohesion** between local particles
- **Alignment** between local particles
- **Separation** between too close particles

=> Leads to emerging global behaviors.



Original video in 1986 (C. Reynolds)

Boids Model - Basic model.

- Set random initial position/speed to N particles.

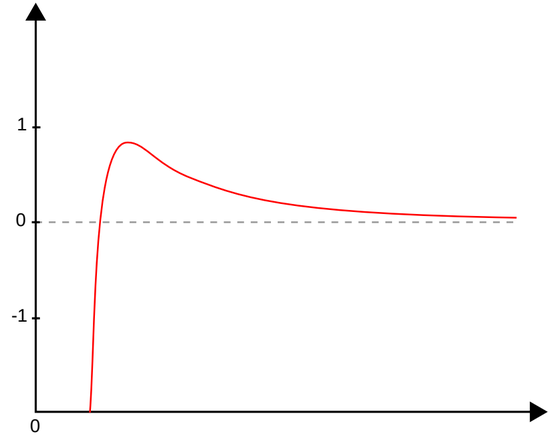
- Set attraction/repulsion force depending on pairwise distances

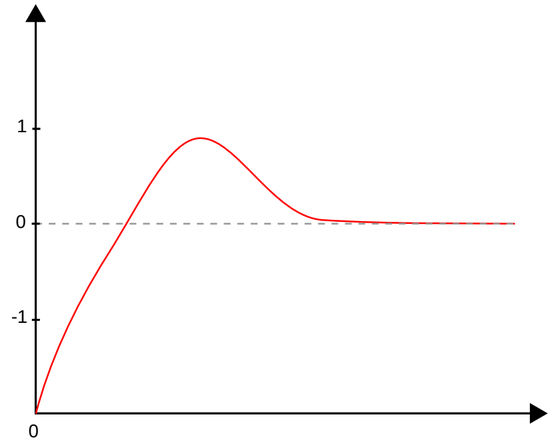
$$F(p_i) = \sum_j f(\|p_j - p_i\|) \frac{p_j - p_i}{\|p_j - p_i\|}$$

Example

- Inverse of distance $f(x) = \frac{\alpha_1}{x^2} - \frac{\alpha_2}{x^4}$

- Exponential/Gaussian $f(x) = \alpha_1 \exp\left(-k\left(\frac{x - x_0}{x_0}\right)^2\right) - \alpha_2 \exp\left(\frac{x}{x_0}\right)$





- Integrate position and speed through time

$$v^{t+\Delta t}(p_i) = v^t(p_i) + \Delta t \; F(p_i^t)$$
$$p^{t+\Delta t}(p_i) = p^t(p_i) + \Delta t \; v^{t+\Delta t}(p_i)$$

Boids Model - Complexity.

Simple implementation:

```
struct particle { vec3 p, v, f; };
std::vector<particle> boids;
// Initialize N boids ...
// ...



// compute pairwise force
for(int i=0; i<N; ++i)
{
    for(int j=0; j<N; ++j)
    {
        if( i!=j )
        {
            const vec3& pi = boids[i].p;
            const vec3& pj = boids[j].p;
            boids[i].f += force( norm(pi,pj)) / (pi-pj)/norm(pi-pj);
        }
    }
}
// integration
for(int i=0; i<N; ++i)
{
    boids[i].v = boids[i].v + dt * boids[i].f;
    boids[i].p = boids[i].p + dt * boids[i].v;
}
```

- What is the complexity (wrt. N) of this algorithm ?
- Can you think of a way to be more efficient for large N ?

Boids Model - Usage and limitations

- Well adapted to flocks (birds, fishes - looking behavior)
- Display particles using 3D animated model

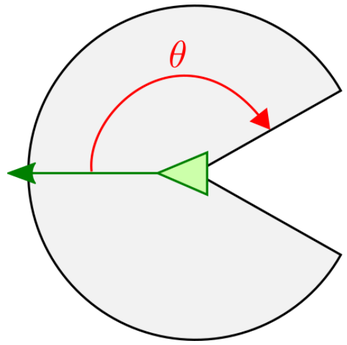
Additional behaviors

- Objective position/speed value
- Constraints: Obstacle avoidance, limited velocity
- Pursue and evade target/other particles - follow the leader, predators, etc.

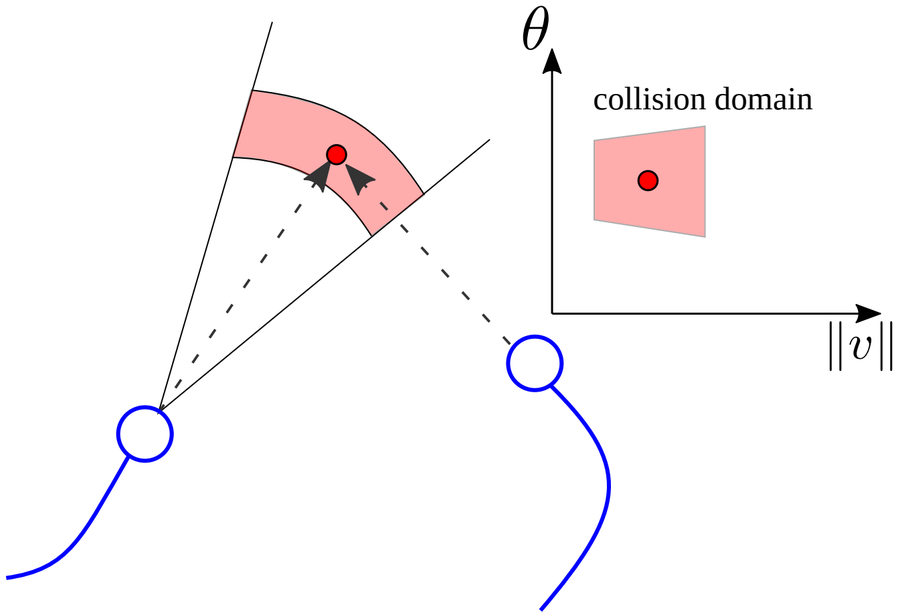
- *Is collision between particles possible ?*
- *Human displacement are mostly guided by vision, what key element is missing in the basic boids force-based model ?*

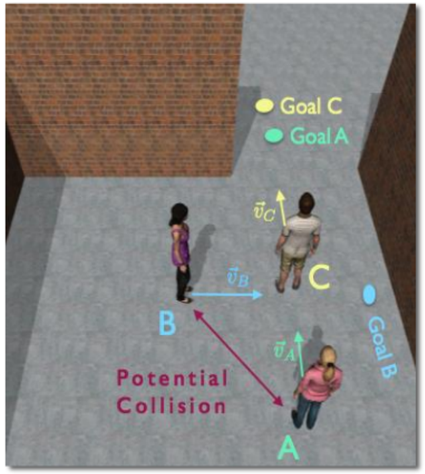
Boids Model - Extensions

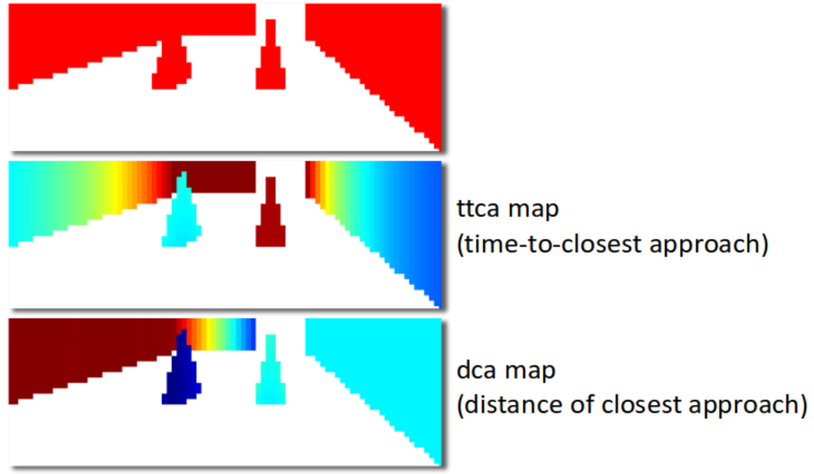
Improving force computation
- Vision-based force computation (angle, speed) : limited view angle



Improving collision avoidance
- Velocity-based
- Vision-based





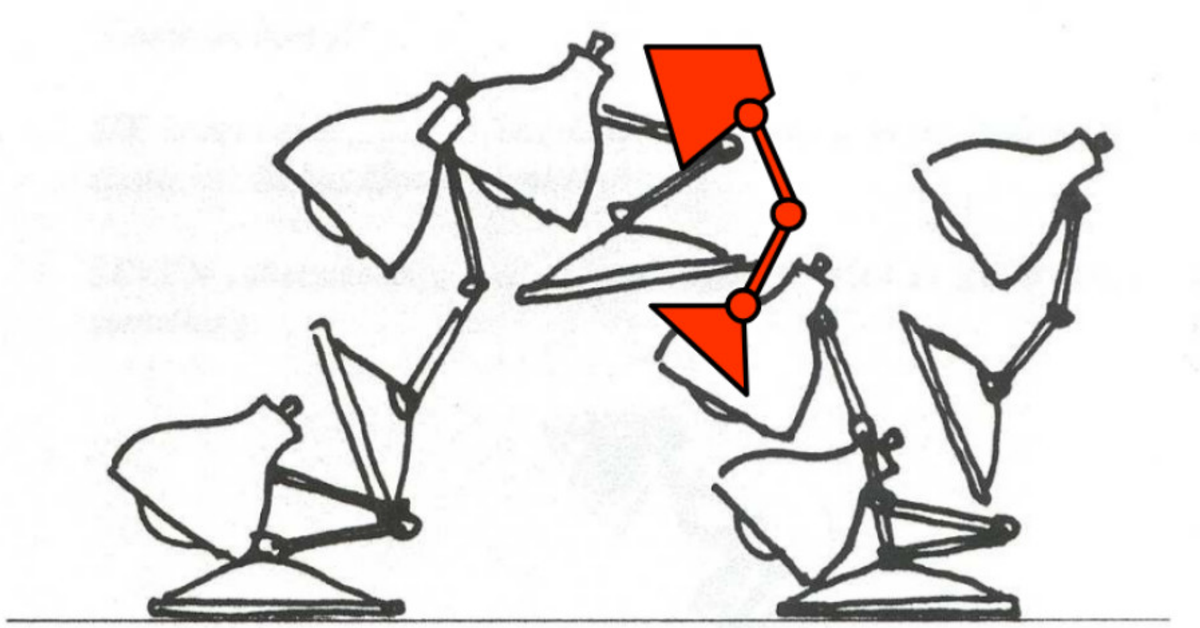


Julien Pettré, Inria Rennes

[J. Ondrej et al. A synthetic-vision based steering approach for crowd simulation, SIGGRAPH 2010]

Keyframe interpolation

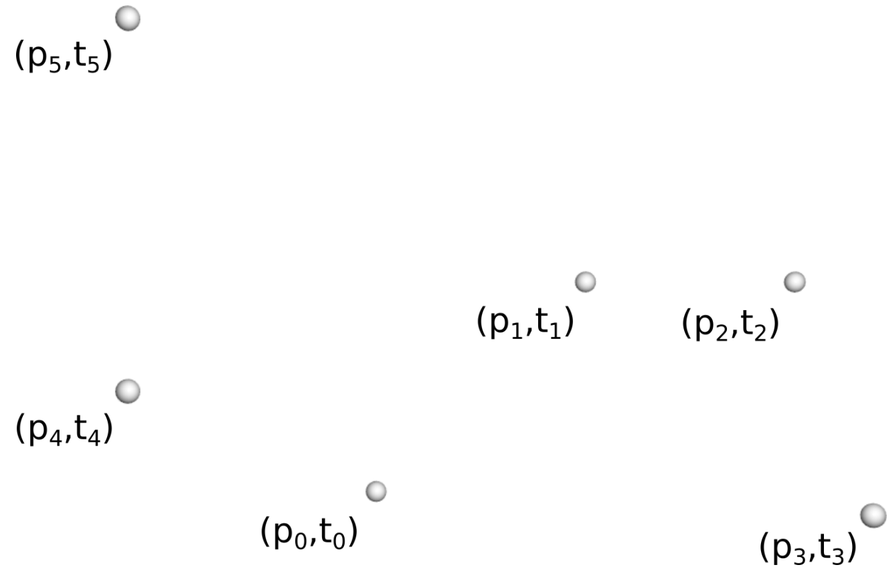
- **Interpolate positions**
- Interpolate rotations



Interpolate positions

Objective

- Given a set of key-positions (positions+time) we want to find an interpolating space-time curve



- **Input** $(p_i, t_i), i \in [0, N-1]$
- **Output**
  - Space time curve $p(t), t \in [t_0, t_{N-1}]$
  - Space time curve $p(t_i) = p_i$

Linear Interpolation

- Simplest solution: linear interpolation between each sample pairs

- $\forall t \in [t_i, t_{i+1}]$, $\begin{cases} p(t) = (1 - \alpha(t))\, p_i + \alpha(t)\, p_{i+1} \\ \alpha(t) = \dfrac{t - t_i}{t_{i+1} - t_i} \end{cases}$

Pros

- Simple
- Constant speed between keyframes

  Easy to ajust

Cons

- Non smooth trajectory
- Generates straight segments

  Artists prefer non straight trajectories for "real" looking motion

Smooth curve

Objective: Generate a **smooth** interpolating space-time curve

Classical Idea Use polynomials curves

$$\begin{cases} p(t) = \sum_{i=0}^{N-1} \alpha_i(t)\, p_i \\ \alpha_i(t) = \sum_{j=0}^{d} c_i^j\, t^i \end{cases}$$

(α_i) polynomial basis function of degree d

Which polynomials/degree choose ?

Lagrange polynomial interpolation

Naive idea: Interpolate all points at once

$$\forall t \in [t_0, t_{N-1}]\ ,\ \ p(t) = \sum_{i=0}^{N-1} \alpha_i(t)\ p_i \qquad \forall i \in [0, N-1]\ \ p(t_i) = p_i$$

Degree of polynomial : $N-1$

- Known solution: **Lagrange polynomial**

$$p(t) = \sum_{i=0}^{N-1} \alpha_i(t)\ p_i \qquad \alpha_i(t) = \prod_{k=0, k\neq i}^{N-1} \frac{t - t_k}{t_i - t_k}$$

- Explanation: By construction $\alpha_i(t_i) = 1$ and $\alpha_i(t_k) = 0$

(+) Interpolate all points

(-) Large oscillations between samples for large degree.

(-) Non local influence

=> Not used in practice

Lagrange polynomial interpolation : Comparison

Ex. Global effect on curve when two samples are added.

10 samples

12 samples

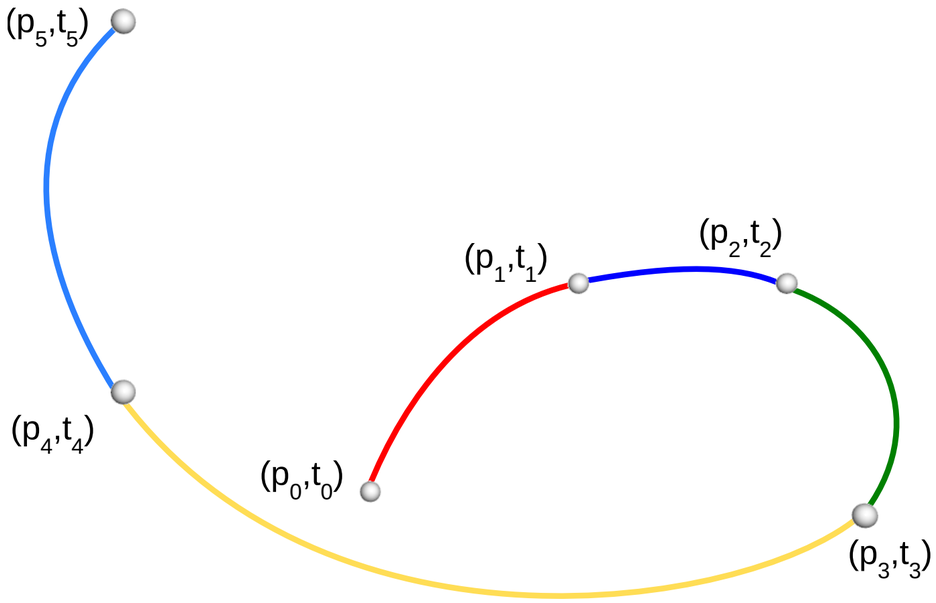
Spline

Idea

- Define on each part a polynomial
- Smooth junctions between them.

How to choose the polynomial

- Sufficiently high degree to be smooth
- Sufficiently low degree to avoid oscillations



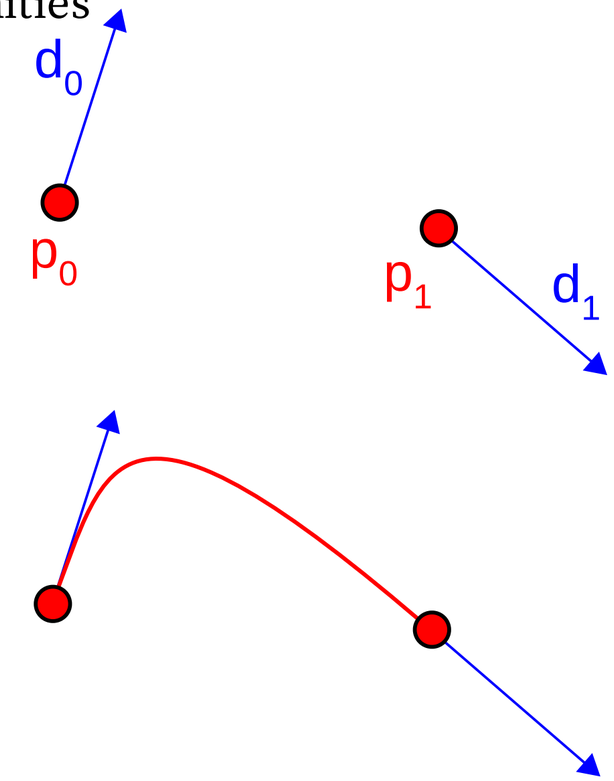
Each color is another polynomial

=> In Graphics cubic polynomials are often used

Allows up to C^2 junctions

Hermite interpolation

Hermite interpolation : cubic curve interpolating points and derivatives at extremities



Consider the following constraints

$$\begin{cases} p(s) = c_3 s^3 + c_2 s^2 + c_1 s + c_0 \\ p(0) = p_0,\ p(1) = p_1,\ p'(0) = d_0,\ p'(1) = d_1 \end{cases}$$

$\Rightarrow$ System of equations

$$\begin{cases} \qquad\qquad\quad c_0 = p_0 \\ c_3 + \quad c_2 + \quad c_1 + \quad c_0 = p_1 \\ \qquad\quad c_1 \qquad\quad = d_0 \\ 3\,c_3 + \quad 2\,c_2 + \quad c_1 \qquad\quad = d_1 \end{cases} \Rightarrow \begin{cases} c_0 = p_0 \\ c_1 = d_0 \\ c_2 = -3p_0 + 3p_1 - 2d_0 - d_1 \\ c_3 = 2p_0 - 2p_1 + d_0 + d_1 \end{cases}$$

$$\forall s \in [0,1]\ ,\quad p(s) = (2s^3 - 3s^2 + 1)\,p_0 + (s^3 - 2s^2 + s)\,d_0 + (-2s^3 + 3s^2)\,p_1 + (s^3 - s^2)\,d_1$$

For arbitrary $t \in [t_i, t_{i+1}]$, we set $s = \dfrac{t - t_i}{t_{i+1} - t_i}$, $\tilde{d}_{0/1} = (t_{i+1} - t_i)\,d_{0/1}$

Interpolating curve

Our initial problem: set of multiple keyframes position+time

Two solutions

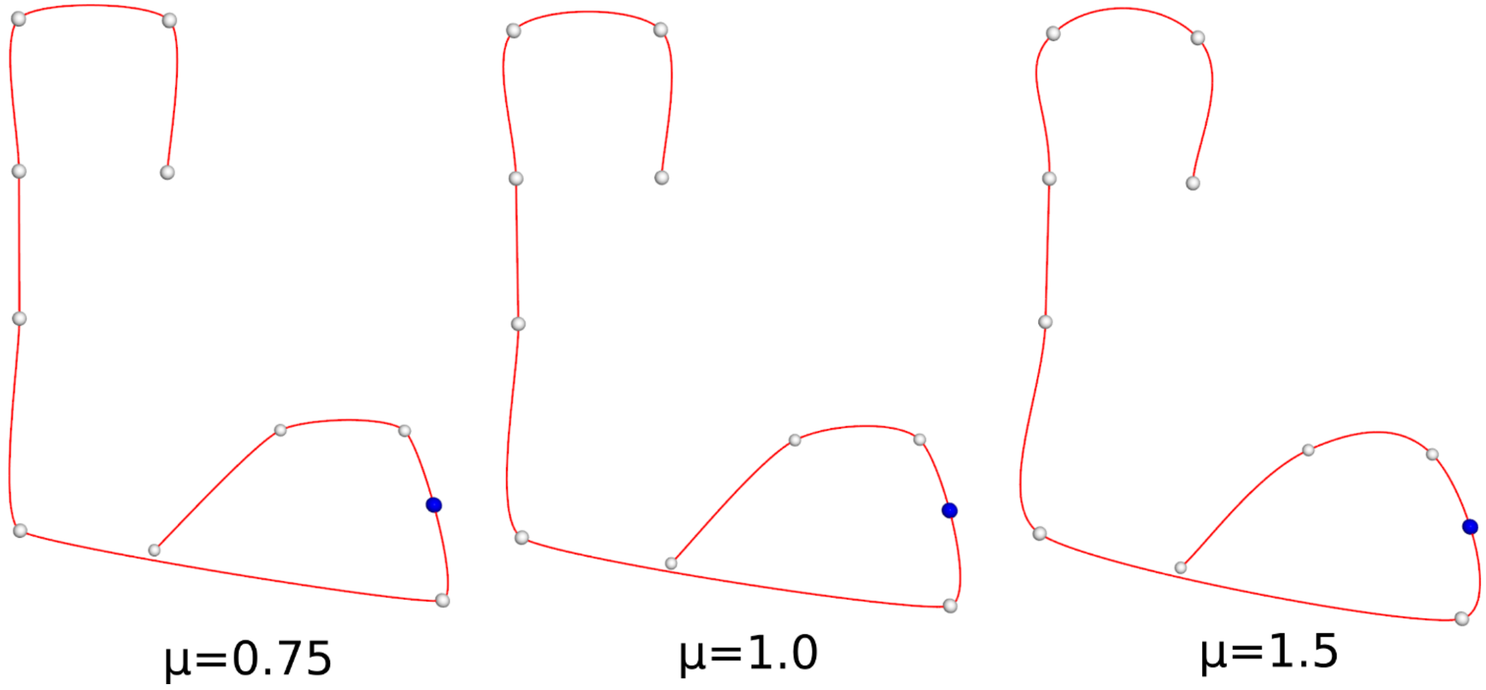
- Set explicitely derivatives for each keyframe - *teadious*
- Compute automatically plausible derivatives from surrounding samples - *often used*

Cardinal spline

Set $d_i = \mu \dfrac{p_{i+1} - p_{i-1}}{t_{i+1} - t_{i-1}}$

- μ curve tension $\in [0, 2]$
- $\mu = 1$ is commonly used

Catmull Rom Spline



Wrap-up Algorithm

Compute $p(t)$ as a cubic spline interpolation

- Given keyframes $(p_i, t_i)_{i \in [0, N-1]}$
- Given time $t \in [t_1, t_{N-2}]$

1. Find i such that $t \in [t_i, t_{i+1}]$
2. Compute $d_i = \mu \, \sigma \, \dfrac{p_{i+1} - p_{i-1}}{t_{i+1} - t_{i-1}}$, and $d_{i+1} = \mu \, \sigma \, \dfrac{p_{i+2} - p_i}{t_{i+2} - t_i}$

   $\sigma = t_{i+1} - t_i$
3. Compute $p(t) = (2s^3 - 3s^2 + 1)\, p_i + (s^3 - 2s^2 + s)\, d_i + (-2s^3 + 3s^2)\, p_{i+1} + (s^3 - s^2)\, d_{i+1}$

   with $s = \dfrac{t - t_i}{\sigma}$.

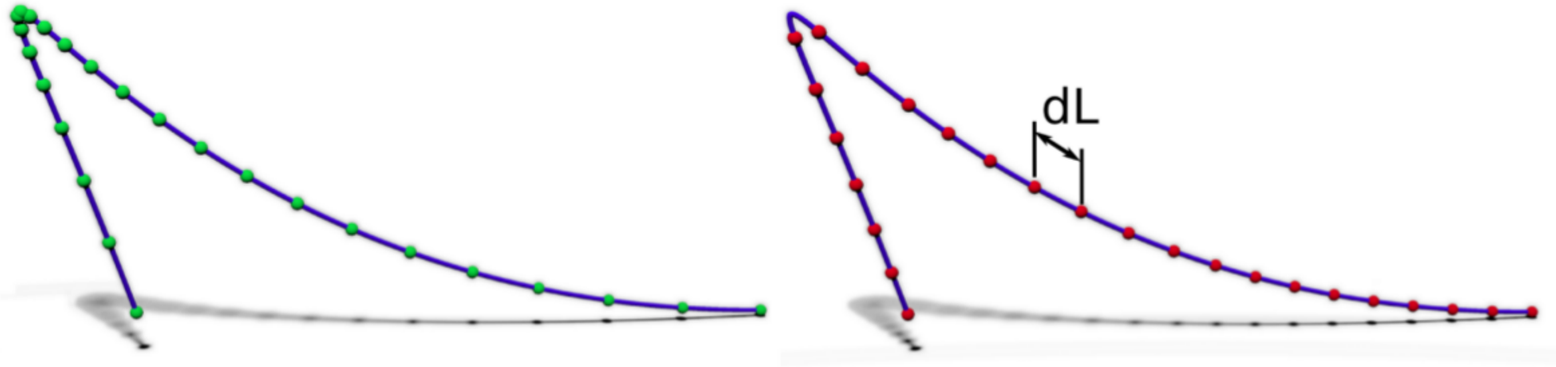
Limitation of cubic curve interpolation

- Only C^1, but not C^2 at junctions : Curvature/acceleration discontinuity

  - *Force C^2 continuity for cubic polynomial (global linear system to solve - loose local structure)*
  - *Consider higher degree polynomial*

- Non constant speed along each polynomial

  - *Reparametrization with curvilinear length*



Usage of keyframes interpolation

Interpolate every vertex of multiple meshes

Multi-target blending and blend shapes

The standard for facial animation

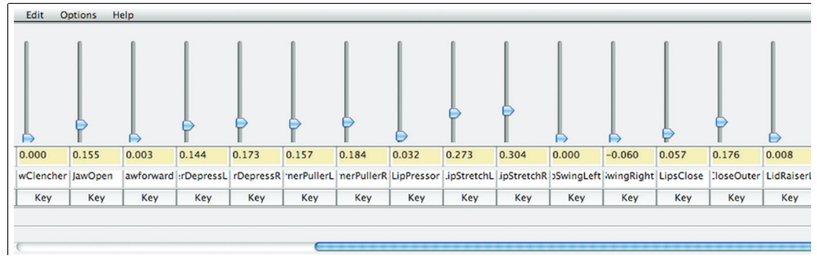
Full blend shape

- Per-vertex formulation $p^i(t) = \sum_k \omega_k(t)\, b_k^i$
- Matrix formulation $\mathbf{p}(t) = \mathrm{B}\,\omega(t)$

Differential representation

- Per-vertex formulation $p^i(t) = b_0^i + \sum_k \omega_k(t)\, (\underbrace{b_k^i - b_0^i}_{d_k^i})$
- Matrix formulation $\mathbf{p}(t) = \mathbf{b_0} + \mathrm{D}\,\omega(t)$

$\Rightarrow$ Allows expression transfert (keep same D for different $\mathbf{b_0}$).



Problems:

- D is not an orthogonal matrix (non unique combination, redundancies) $\Rightarrow$ decomposition PCA, etc.
- How to directly manipulate blend shapes

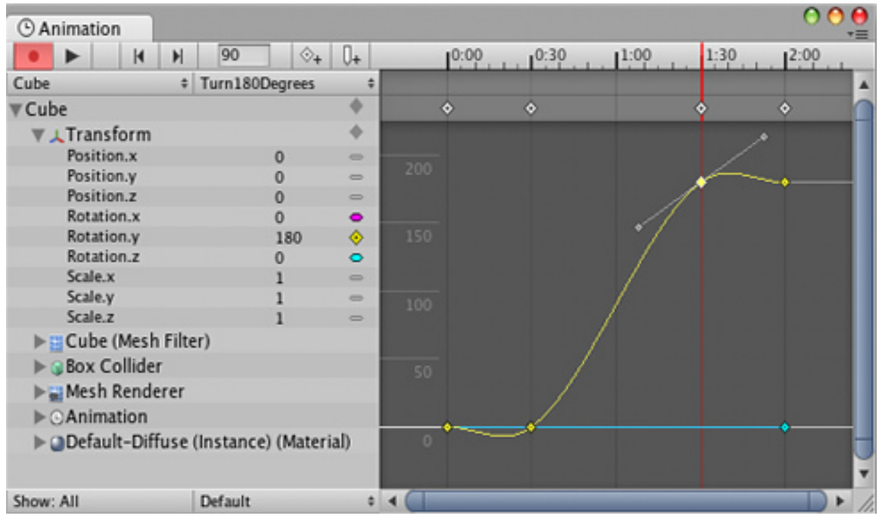
[J.P. Lewis et al. Practice and Theory of Blendshape Facial Models. EG STAR 2014]

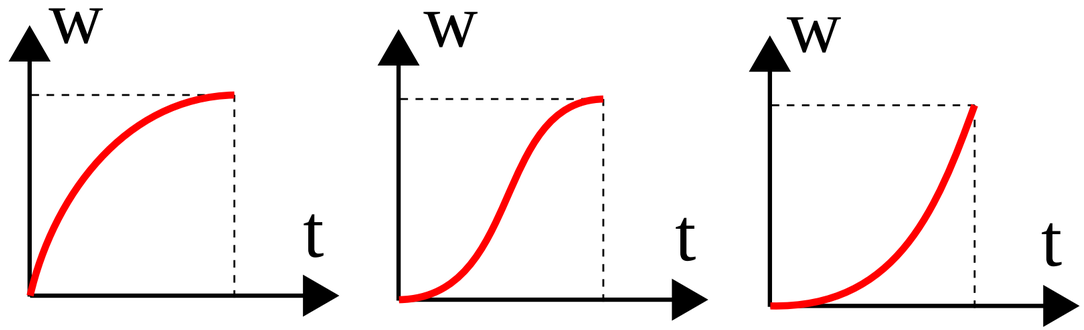
Curve editing

- Animation software (Maya, 3DSMax, Blender, etc) always come with a **curve editor**.

- Artists can manually ajust their position, time, and **derivatives** on curve editor.
  - One curve for each scalar parameter
    - position (x, y, z)
    - scaling (s_x, s_y, s_z)
    - rotation/quaternion (q_x, q_y, q_z, q_w)

- Can also use a wrapper function w to change time $p(t) = f(w(t))$



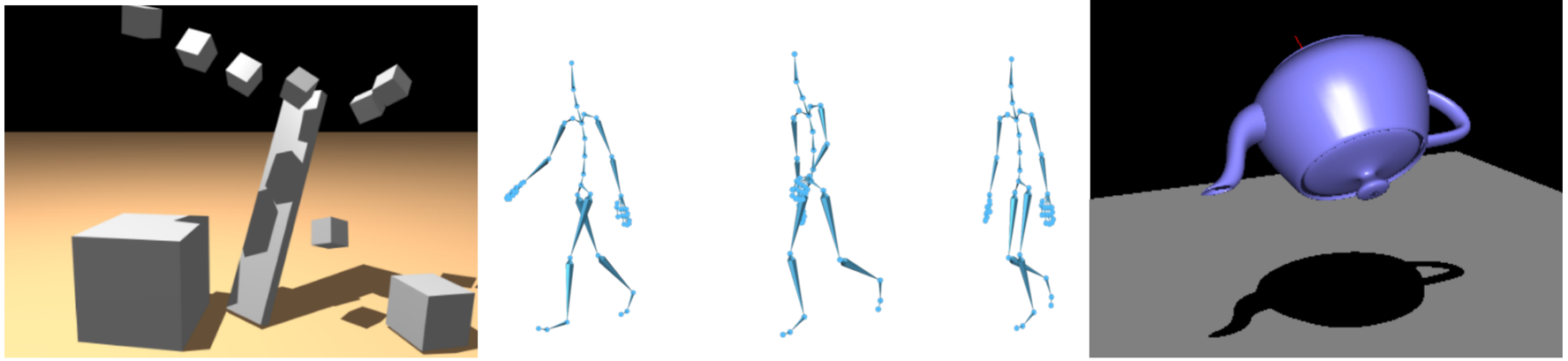


Interpolate rotations

and rigid transformations

Rotation

When dealing with rigid objects (ex. Character skeleton, oriented rigid objects, etc) orientations (thus rotations) are involved.



- How to **encode and interpolate rotations** ?

Rotation in 2D: Easy

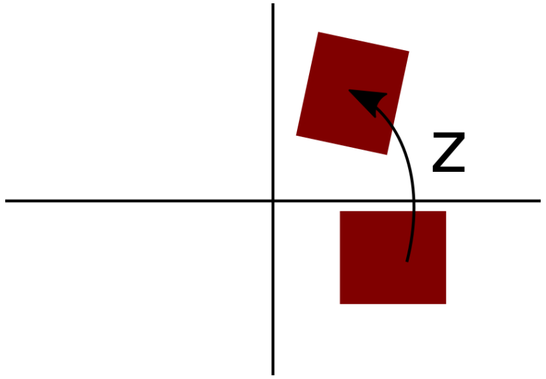
1 - Model rotation by an angle

2 - Interpolate the angle

Can be formalized using complex numbers representation

- $R_1 \rightarrow z_1 = e^{i\,\theta_1}$
- $R_2 \rightarrow z_2 = e^{i\,\theta_2}$

- $R^{\alpha}_{1\rightarrow 2} \rightarrow z = e^{i\,(\alpha\theta_1 + (1-\alpha)\theta_2)}$



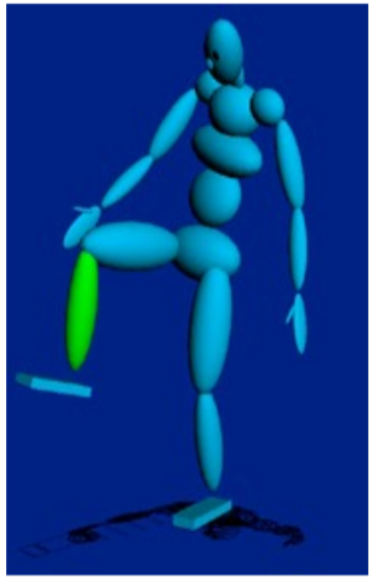
In 3D: not so easy ...

How many angles do I need to represent a 3D rotation?

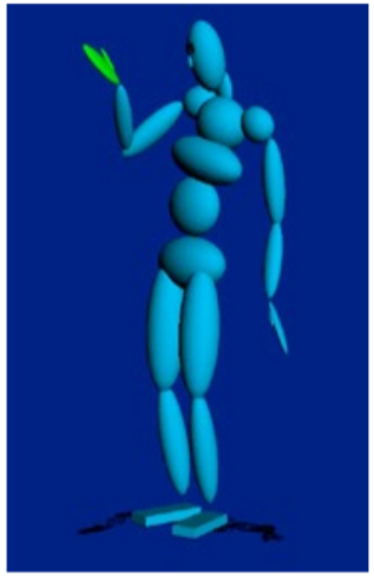
Rotation in 3D - DOF

- 3 Degrees of Freedom (dof)

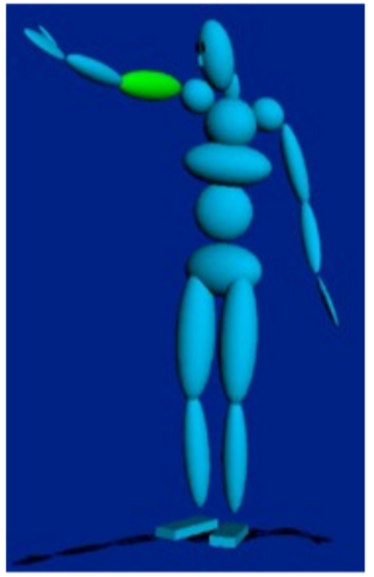
1 DOF: knee



2 DOF: wrist



3 DOF: arm



Representing 3D Rotations

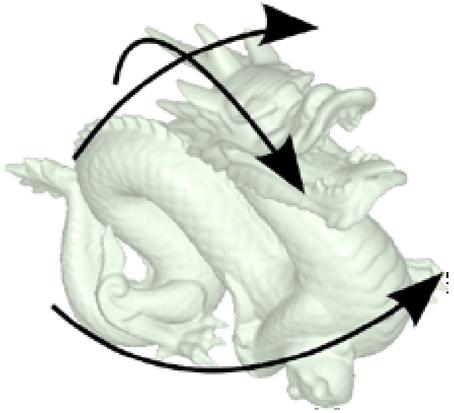
Multiple ways to represent rotations

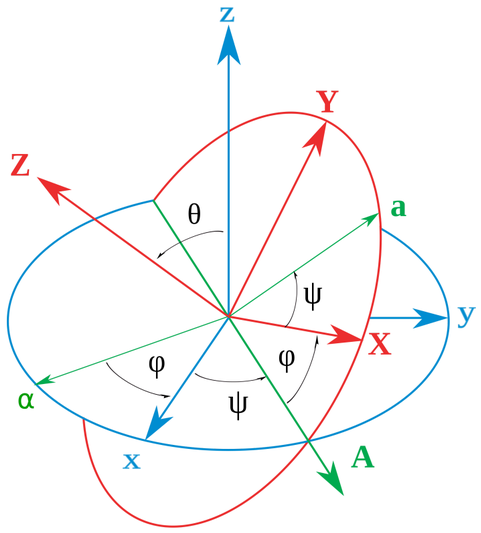
1 - Matrices

2 - Euler (/Trait-Bryan) angles

3 - Axis-Angle

4 - Quaternion





Rotation representation: Matrix

$$\begin{cases} \mathrm{R} = \begin{pmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{pmatrix} \\ \\ \mathrm{R}^T \, \mathrm{R} = \mathrm{Id} \end{cases}$$

Pro.

- Standard structure (storage,manipulate)
- Simple application to vector: $v' = \mathrm{R}v$
- Composition b/w rotations: $\mathrm{R} = \mathrm{R}_1 \, \mathrm{R}_2$

Cons.

- Large redundancy (*9 coefficients for 3 dof*).
- Rotation parameters don't appear explicitely
- Unclear interpolation
  - Linear interpolation doesn't work well for large angles

  ex. $\mathrm{M} = (1 - \alpha) \, \mathrm{R}_1 + \alpha \mathrm{R}_2$

  => M is not a rotation anymore (not a vectorial space)

Representing 3D Rotations

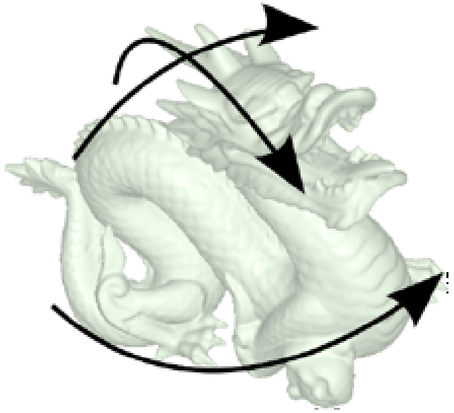
Multiple ways to represent rotations

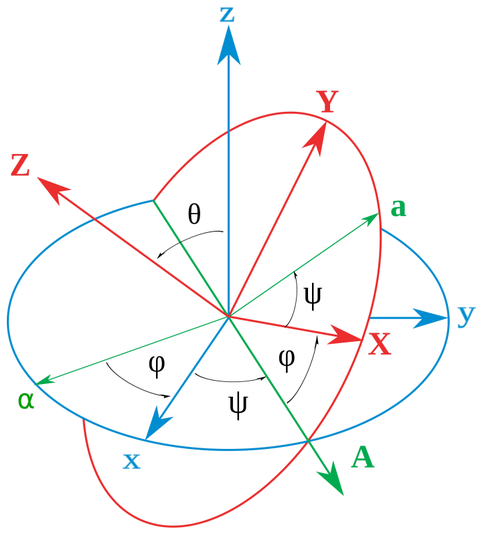
1 - Matrices

2 - Euler (/Trait-Bryan) angles

3 - Axis-Angle

4 - Quaternion





Rotation representation: Euler angles

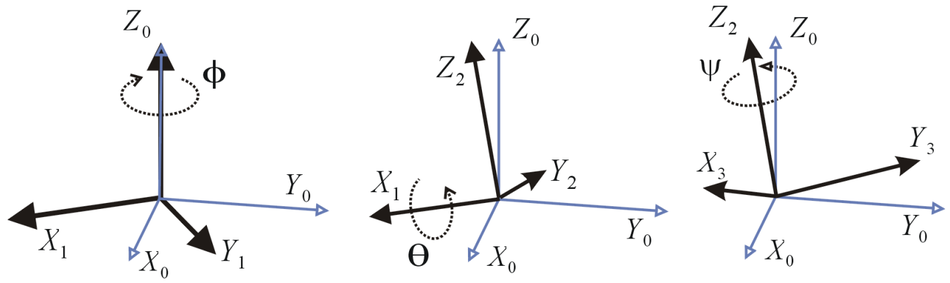
Three consecutive rotations along (x, y, z) coordinates.

Can use basic rotation matrices

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin\theta & \cos(\theta) \end{pmatrix} \quad R_y = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \quad R_z = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

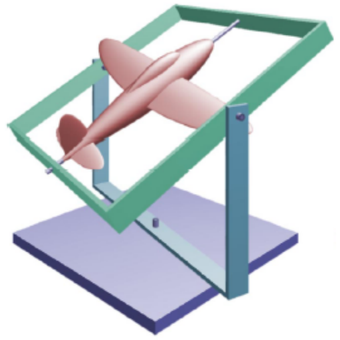
Multiple Euler angles conventions

- *Proper Euler : z-x-z', x-y-x', y-z-y', ...*
- *Trait-Bryan : x-y-z (global), ...*



Pro

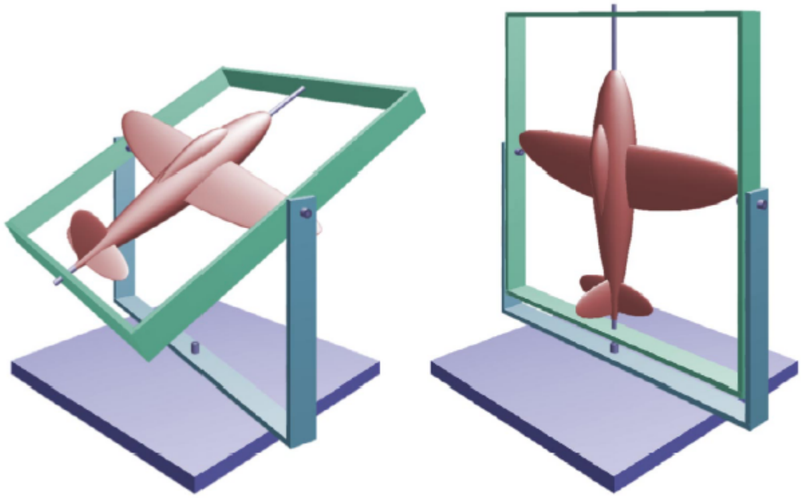
- Combination of rotation around known axis
- Comprehensive parameters (3 dof)
- Animators can interact with angular curves
- Easy conversion to matrix
- *Widely used in robotics*



Rotation representation: Euler angles

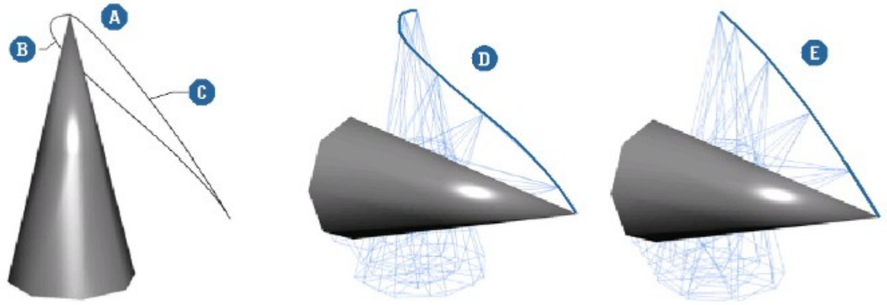
Limitations of Euler Angle

- *Gimbal Lock* when composing b/w some rotations



http://www.fho-emden.de/~hoffmann/gimbal09082002.pdf

- Interpolation of 3 angles leads to non-intuitive trajectory.



Representing 3D Rotations

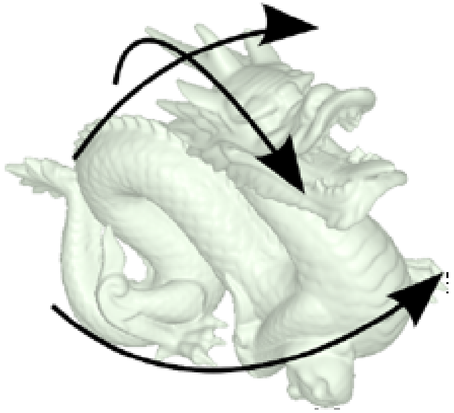
Multiple ways to represent rotations

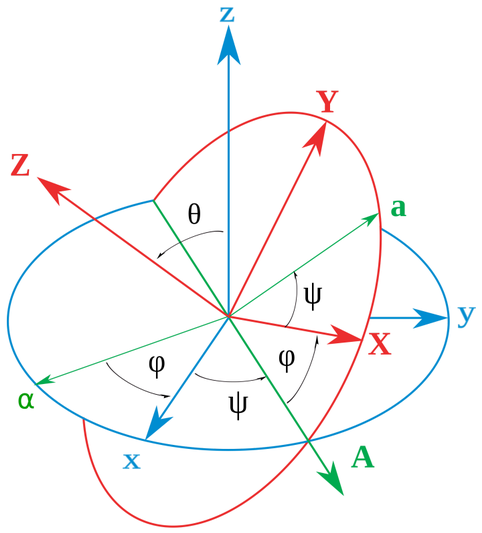
1 - Matrices

2 - Euler (/Trait-Bryan) angles

3 - Axis-Angle

4 - Quaternion

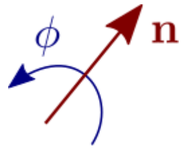




Rotation representation: Axis-Angle

Any 3D rotation can be represented by

- A unit axis n
- An angle θ

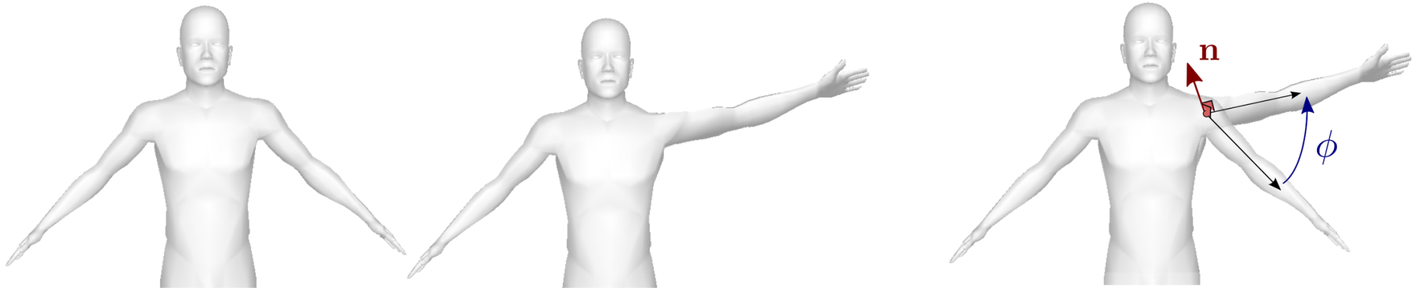


Pro.

- Concise representation: 3 DOF
- Meaningfull parameters
- Describes well the rotation between two vectors

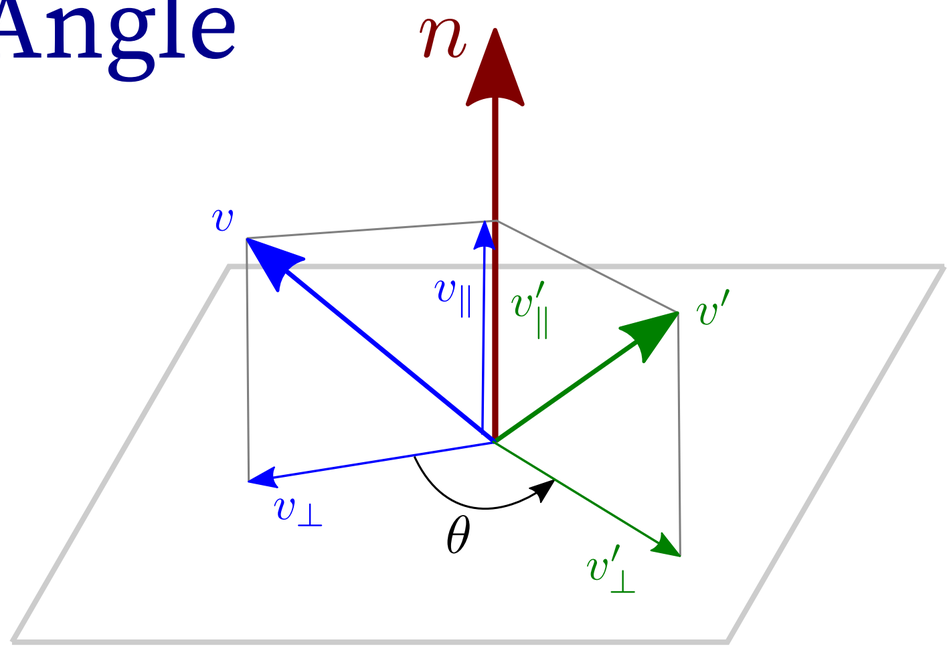
Ex. Rotation between u_1 and u_2

- Axis of rotation $n = (u_1 \times u_2)/\|u_1 \times u_2\|$
- Angle of rotation $\theta = \text{acos}(u_1 \cdot u_2)$
- Twist around u_2 can be arbitrarily chosen



Rotation representation: Axis-Angle

Applying a rotation (n, θ) to a vector v



$$v = v_\parallel + v_\perp$$
$$v' = v'_\parallel + v'_\perp$$
$$\Rightarrow v' = v_\parallel + (\cos(\theta)\, v_\perp + \sin(\theta)\, n \times v_\perp)$$

$$v_\parallel = (v \cdot n)\, n$$
$$v_\perp = v - (v \cdot n)\, n$$

$$v' = (v \cdot n)\, n + \cos(\theta)(v - (v \cdot n)\, n) + \sin(\theta)\, n \times (v - (v \cdot n)\, n)$$

$$\Rightarrow v' = \cos(\theta)\, v + \sin(\theta)\, n \times v + (1 - \cos(\theta))\, (v \cdot n)\, n$$

Rodrigues' rotation Formula

Axis-Angle as matrix

$$v' = \cos(\theta)\, v + \sin(\theta)\, n \times v + (1 - \cos(\theta))\, (v \cdot n)\, n = R(n, \theta)\, v$$

$$v' = \cos(\theta) v + \sin(\theta) \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \times v + (1 - \cos(\theta)) \begin{pmatrix} n_x & n_y & n_z \end{pmatrix} v \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

$$v' = \cos(\theta)\, v + \sin(\theta) \underbrace{\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}}_{\mathrm{K}} v + (1 - \cos(\theta)) \underbrace{\begin{pmatrix} n_x^2 & n_x\, n_y & n_x\, n_z \\ n_x\, n_y & n_y^2 & n_y\, n_z \\ n_x\, n_z & n_y\, n_z & n_z^2 \end{pmatrix}}_{\mathrm{K}^2} v$$

$$v' = \begin{pmatrix} \cos(\theta) + n_x^2(1 - \cos(\theta)) & n_x n_y (1 - \cos(\theta)) - n_z \sin(\theta) & n_x n_z (1 - \cos(\theta)) + n_y \sin(\theta) \\ n_x n_y (1 - \cos(\theta)) + n_z \sin(\theta) & \cos(\theta) + n_y^2 (1 - \cos(\theta)) & n_y n_z (1 - \cos(\theta)) - n_x \sin(\theta) \\ n_x n_z (1 - \cos(\theta)) - n_y \sin(\theta) & n_y n_z (1 - \cos(\theta)) + n_x \sin(\theta) & \cos(\theta) + n_z^2 (1 - \cos(\theta)) \end{pmatrix} v$$

Rotation representation: Axis-Angle

Given a rotation (n, θ) - Corresponding rotation matrix

$$R(n,\theta) = \mathrm{I} + \sin(\theta)\,\mathrm{K} + (1-\cos(\theta))\,\mathrm{K}^2$$

$$R(n,\theta) = \begin{pmatrix} \cos(\theta) + n_x^2(1-\cos(\theta)) & n_x n_y(1-\cos(\theta)) - n_z\sin(\theta) & n_x n_z(1-\cos(\theta)) + n_y\sin(\theta) \\ n_x n_y(1-\cos(\theta)) + n_z\sin(\theta) & \cos(\theta) + n_y^2(1-\cos(\theta)) & n_y n_z(1-\cos(\theta)) - n_x\sin(\theta) \\ n_x n_z(1-\cos(\theta)) - n_y\sin(\theta) & n_y n_z(1-\cos(\theta)) + n_x\sin(\theta) & \cos(\theta) + n_z^2(1-\cos(\theta)) \end{pmatrix}$$

Pro

- Concise and general representation
- Expressive parameters in 3D space (axe, angles)
- Efficient rotation and correspondance with matrix

Cons

- No simple composition expression between two rotations.
- No direct interpolation

Composition between axis angle representation

Given two rotations (n_1, θ_1), (n_2, θ_2)

The composition $(n_3, \theta_3) = (n_1, \theta_1) \circ (n_2, \theta_2)$ is expressed by

- $\cos\left(\frac{\theta_3}{2}\right) = \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) - \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) n_1 \cdot n_2$

- $n_3 = \frac{1}{\tan\left(\frac{\theta_3}{2}\right)} \frac{\tan\left(\frac{\theta_2}{2}\right) n_2 + \tan\left(\frac{\theta_1}{2}\right) n_1 - \tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right) n_1 \times n_2}{1 - \tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right) n_1 \cdot n_2}$

- Can be derived using quaternions
- Rare to use the formula as it is

Representing 3D Rotations

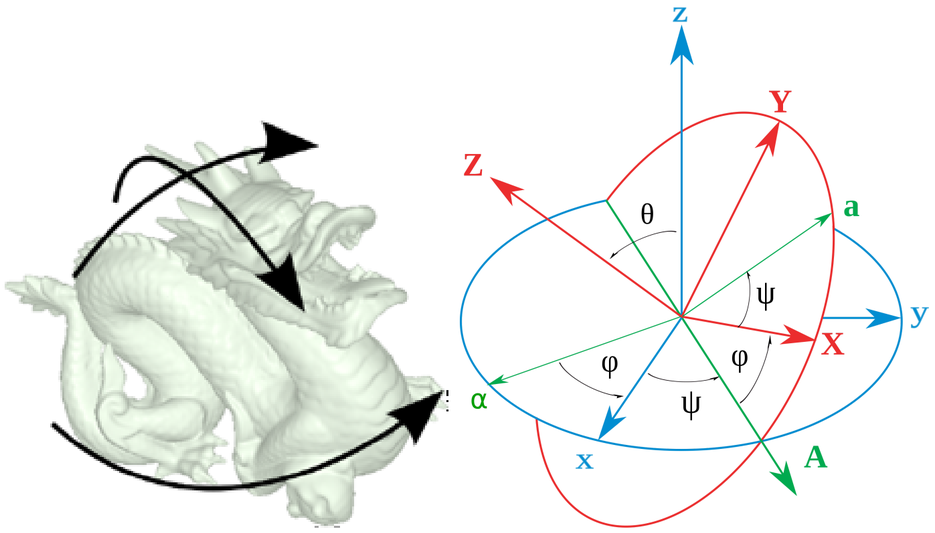
Multiple ways to represent rotations

1 - Matrices

2 - Euler (/Trait-Bryan) angles

3 - Axis-Angle

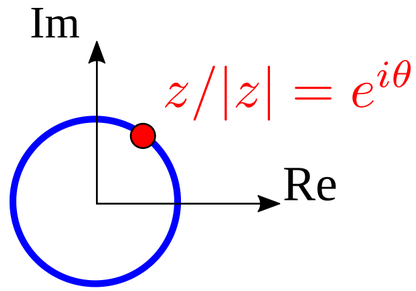
4 - Quaternion



Intuition for quaternion

Going back to 2D and complex space

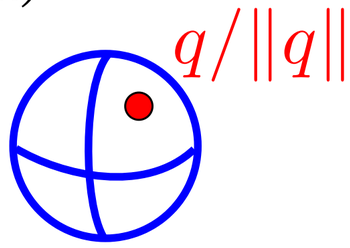
- 2D rotations have 1 dof. (rotation angle)
- 2D rotation represented as a point on a unit circle (S^1).
  - *1D structure embedded in a 2D space.*
  - *Complex space has algebraic structure adapted to model 2D rotation.*



rotations
sphere S^2

Moving to 3D

- 3D rotations have 3 dof.
  - *(point on a unit 3D sphere allows only 2 dof : not sufficient ex. orientation that can twist)*
  - point on a unit 4D sphere allows 3 dof
- 3D rotation are represented as **point on the unit sphere in 4D** (S^3).
  - *3D structure embedded in a 4D space.*
  - *Complex quaternion space has algebraic structure adapted to model 3D rotation.*



rotations
sphere S^4

Big picture of quaternions

Share similarities with 2D complex

- Complex value object with 4 components $q = (x, y, z, w)$
- Unit quaternions represent rotations $q/\|q\|$.
- Quaternion product represent rotation composition $q_1\, q_2$

Some differences

- 3D vectors are points in pure quaternion subspace $q_v = (x, y, z, 0)$
- Applying quaternion to points $q_{v'} = q\, q_v\, q^\star$

Advantage of quaternion

- Interpolation works well in quaternion space (interpolation of points on sphere).

Rotation representation: Quaternions

Quaternions: generalization of complex numbers.

$q = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} + w$ w real part, (x, y, z) imaginary (or pure quaternion) part.

We write in short $q = (x, y, z, w)$

(don't confound with 4D vectors in homogeneous coordinates)

Properties of imaginary basis vectors

$$\begin{cases} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k} \\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i} \\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \\ \mathbf{ijk} = -1 \end{cases}$$

- Provides the algebraic properties

Basics operations on quaternions

- Conjugated quaternion $q^\star = (-x, -y, -z, w)$

- Quaternion norm $\|q\| = \sqrt{q\, q^\star} = \sqrt{x^2 + y^2 + z^2 + w^2}$. Unit quaternion satisfies $\|q\| = 1$.

- Quaternion product

$$q_1\ q_2 = (x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k} + w_1)\,(x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k} + w_2) = \dots$$

$$q_1\ q_2 = \begin{pmatrix} x_1 w_2 + w_1 x_2 + y_1 z_2 - z_1 y_2 \\ y_1 w_2 + w_1 y_2 + z_1 x_2 - x_1 z_2 \\ z_1 w_2 + w_1 z_2 + x_1 y_2 - y_1 x_2 \\ w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 \end{pmatrix}$$

Note: Quaternion product is
- associative: $(q_1 q_2) q_3 = q_1 (q_2 q_3) = q_1\ q_2\ q_3$
- **non-commutative**: $q_1\ q_2 \neq q_2\ q_1$

Sometimes interesting to separate *real part* w from *pure quaternion* part $s = (x, y, z)$.
- Shorthand vector form $q = (s, w)$
- Quaternion product in vector form $q_1\ q_2 = (s_1\, w_2 + s_2\, w_1 + s_1 \times s_2\ ,\ w_1\, w_2 - s_1 \cdot s_2)$

Relation between quaternion and rotation

Consider

- a quaternion $q = (s, w)$ of unit norm $\|q\| = 1$.
- a vector $v = (v_x, v_y, v_z)$ assimilated to the pure quaternion $q_v = (v, 0) = (v_x, v_y, v_z, 0)$.

Then

- $q_{v'} = \mathcal{R}_q(v) = q\, q_v\, q^\star$ is a pure quaternion $q_{v'} = (v'_x, v'_y, v'_z, 0)$
- And $v' = (v'_x, v'_y, v'_z)$ is the rotation of vector v around the axis $n = s/\|s\|$, with angle $2\,\mathrm{acos}(w)$.

Demonstration

$\mathcal{R}_q(v) = (s, w)\ (v, 0)\ (-s, w) = \ldots = ((w^2 - s^2)\, v + 2(s \cdot v)\, s + 2w\, (s \times v), 0)$

As $\|q\| = 1$, we can write $q = (s, w) = (n \sin(\phi), \cos(\phi))$, where $\|n\| = 1$

Then $\mathcal{R}_q(v) = (\underbrace{(\cos^2(\phi) - \sin^2(\phi))}_{\cos(2\phi)}\, v + \underbrace{2 \sin^2(\phi)}_{1-\cos(2\phi)} (n \cdot v)\, n + \underbrace{2\, \cos(\phi) \sin(\phi)}_{\sin(2\phi)}\, n \times v\,,\ 0)$

$\Rightarrow$ Rodrigues formula for axis n and angle 2ϕ.

The unit quaternion $q = (n\, \sin(\theta/2), \cos(\theta/2))$ represents the rotation of angle θ around the axis n.

Composition of rotations

Consider two rotation (R_1, R_2) associated to their unit quaternions (q_1, q_2).

The product $q_1\, q_2$ represents the composition $R_1 \circ R_2$.

Demonstration

We show that $\mathcal{R}_{q_1\, q_2}(v) = \mathcal{R}_{q_1} \circ \mathcal{R}_{q_2}(v)$.

$$\mathcal{R}_{q_1\, q_2}(v) = (q_1 q_2)\; v\; (q_1 q_2)^\star$$

$$\mathcal{R}_{q_1\, q_2}(v) = (q_1\; q_2)\; v\; (q_2^\star\; q_1^\star), \text{ as } (q_1 q_2)^\star = q_2^\star\; q_1^\star$$

$$\mathcal{R}_{q_1\, q_2}(v) = q_1\; (q_2\; v\; q_2^\star)\; q_1^\star$$

$$\mathcal{R}_{q_1\, q_2}(v) = q_1\; \mathcal{R}_{q_2}(v)\; q_1^\star = \mathcal{R}_{q_1} \circ \mathcal{R}_{q_2}(v)$$

Correspondance quaternion to rotation matrix

The unit quaternion $q = (x, y, z, w)$ represents the rotation given by the matrix

$$R = \begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{pmatrix}$$

Demonstration

$$v' = \mathcal{R}_q(v) = q\, q_v\, q^\star = ((w^2 - s^2)\, v + 2(s \cdot v)\, s + 2w\, (s \times v), 0) \quad \text{with } s = (x, y, z)$$

$$v' = (w^2 - x^2 - y^2 - z^2)v + 2 \begin{pmatrix} x & y & z \end{pmatrix} v \begin{pmatrix} x & y & z \end{pmatrix}^T + 2w \begin{pmatrix} x & y & z \end{pmatrix} \times v$$

$$v' = \left((w^2 - x^2 - y^2 - z^2)\, \mathrm{I} + 2 \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} + 2w \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \right) v$$

$$v' = R\, v, \text{ with } x^2 + y^2 + z^2 + w^2 = 1$$

Summary - Correspondance quaternion / matrix-vector

Representation and Operations

	3D space	**Quaternion space**
Vector	$v = (v_x, v_y, v_z)$	$q_v = (v, 0) = (v_x, v_y, v_z, 0)$
Rotation	R (3×3 matrix)	$q = (x, y, z, w)$, $\|q\| = 1$
Apply rotation to vector	Rv	$q\ q_v\ q^\star$
Rotation composition	$R_1\ R_2$	$q_1\ q_2$

Rotation to Quaternion

Rotation of axis n and angle $\theta \Rightarrow q = (n \sin(\theta/2), \cos(\theta/2))$

Quaternion to Rotation

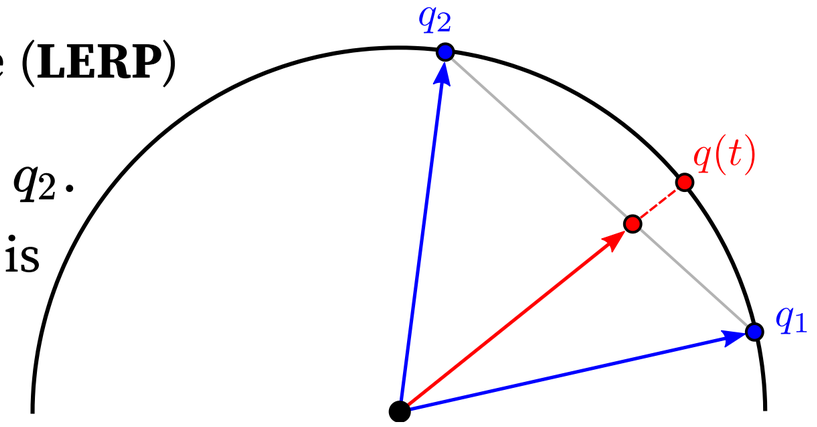
$$\text{Unit quaternion } q = (x, y, z, w) \Rightarrow R = \begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{pmatrix}$$

Interpolation of Quaternion - LERP

Linear interpolation (with normalization) in quaternion space (**LERP**)

- Consider two rotations with corresponding quaternion q_1, q_2.
- The *linearly interpolated* quaternion at parameter $t \in [0, 1]$ is

$$q(t) = \frac{(1-t)\,q_1 + t\,q_2}{\|(1-t)\,q_1 + t\,q_2\|}$$



(+) Follows a shortest path (great circle on 4D sphere)

(+) Can be generalized to more general parametric curves (Splines, etc)

(-) Angular speed of orientation is not-constant

ex. extreme case of two opposite quaternions

Interpolation of Quaternion - SLERP

Spherical Linear interpolation (**SLERP**)

- Consider two rotations with corresponding quaternion q_1, q_2.
- The *spherical linear interpolated* quaternion at parameter $t \in [0, 1]$ is

$$q(t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)} q_1 + \frac{\sin(t\Omega)}{\sin(\Omega)} q_2 \quad \text{with } \cos(\Omega) = q_1 \cdot q_2$$

Demonstration

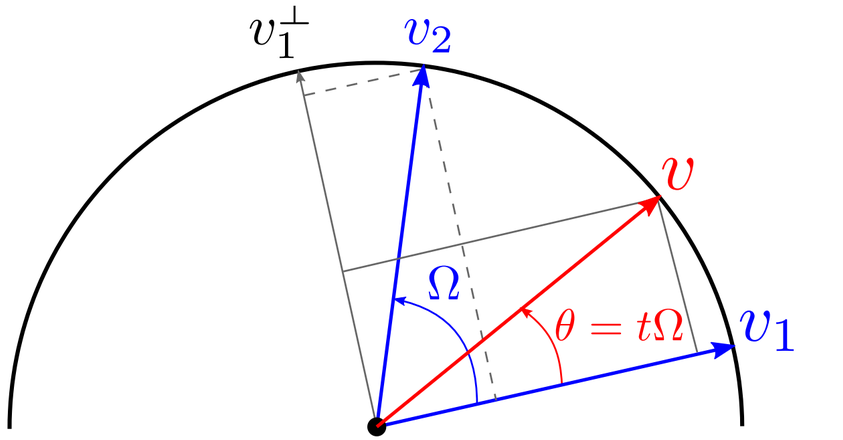
Consider

- Two unit vectors (arbitrary dimensions) v_1, v_2.
- Ω: angle b/w v_1 and v_2
- The interpolated vector v at angle $\theta = \Omega t, t \in [0, 1]$.

$v = v_1 \cos(\theta) + v_1^{\perp} \sin(\theta)$, and $v_1^{\perp} = \frac{v_2 - \cos(\Omega)\, v_1}{\sin(\Omega)}$

$\Rightarrow v = \left(\frac{\sin(\Omega)\ \cos(\theta) - \cos(\Omega)\ \sin(\theta)}{\sin(\Omega)} \right) v_1 + \frac{\sin(\theta)}{\sin(\Omega)} v_2$

$\Rightarrow v = \frac{\sin(\Omega - \theta)}{\sin(\Omega)} v_1 + \frac{\sin(\theta)}{\sin(\Omega)}$

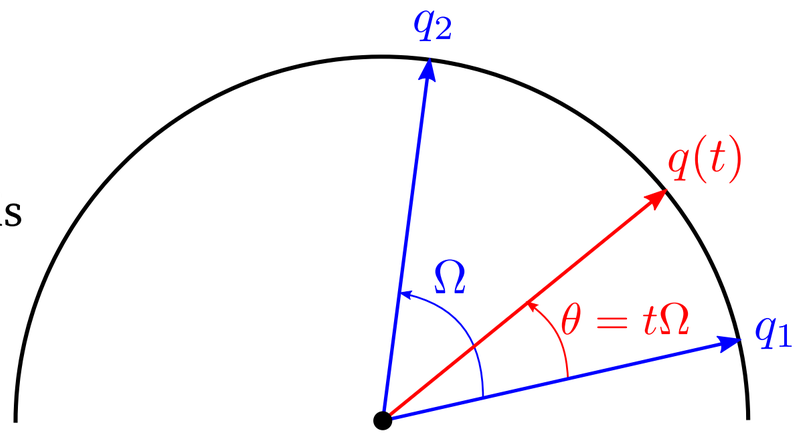


Interpolation of Quaternion - SLERP

Spherical Linear interpolation (**SLERP**)

- Consider two rotations with corresponding quaternion q_1, q_2.
- The *spherical linear interpolated* quaternion at parameter $t \in [0, 1]$ is

$$q(t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)} q_1 + \frac{\sin(t\Omega)}{\sin(\Omega)} q_2 \quad \text{with } \cos(\Omega) = q_1 \cdot q_2$$



(+) Follows a shortest path (great circle on 4D sphere)

(+) Constant angular speed

(-) Cannot be generalized to more general curves.

Cannot interpolates between more than two quaternions

Care with quaternion negation

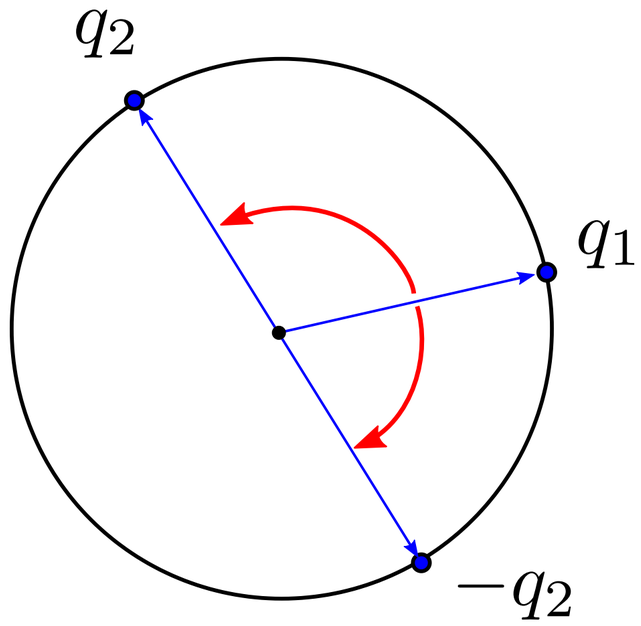
$+q$ and $-q$ correspond to the same rotation $(n \to -n, \theta \to 2\pi - \theta)$

Warning $-q$ ***does not*** *corresponds to the rotation matrix $-R$.*

But to a **different path** when interpolated in the 4D quaternion space.

$\Rightarrow$ path $q_1 \to -q_2$ is shorter than $q_1 \to q_2$ when $q_1 \cdot q_2 < 0$.

In practice we check for the shorter path before applying SLERP.



Algorithm

```
if( dot(q1,q2)<0 )
    q2 = -q2
q(t) = SLERP(q1,q2,t)
```

Interpolating rigid motion

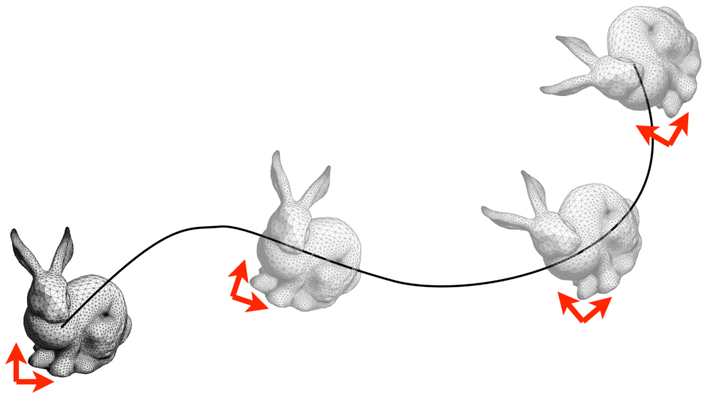
value 0
Close Controls

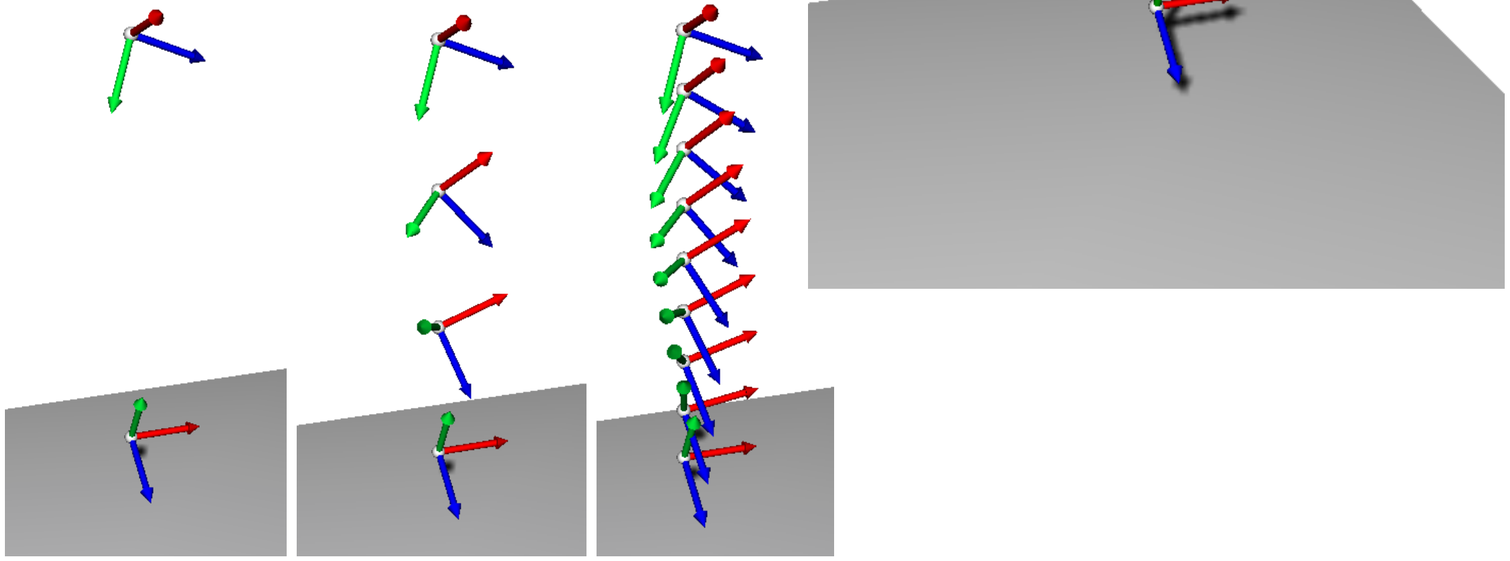
3D objects have orientation r and position p.

$\Rightarrow$ Need to interpolate both, usually handled separately.

Given two key-positions (p_1, r_1) and (p_2, r_2) *in-betweens* computed at time $t \in [0, 1]$ as

- Linear interpolation of positions $p(t) = (1 - t)\, p_1 + t\, p_2$
- Interpolate rotation with SLERP on quaternions
  - Convert $(r_1, r_2) \rightarrow (q_1, q_2)$
  - Compute $q(t) = SLERP(q_1, q_2, t)$
  - Convert back $q(t) \rightarrow r(t)$





Interpolating rigid motion - Comparison

Matrix interpolation

Euler angle interpolation

Quaternion interpolation

Handling affine transformation : Polar Decomposition

Key-pose can be given as matrix M containing rotation, but also scaling and shearing.

$\Rightarrow$ Cannot convert M to quaternion (not a rotation).

$\Rightarrow$ Separate *rotation* component from shearing and scaling component using **polar decomposition**.

[K. Shoemake and T. Duff, Matrix Animation and Polar Decomposition. Graphics Interface 92.]

- Polar decomposition: $\mathrm{M} = \mathrm{R}\,\mathrm{D}$
  - R: Rotation matrix
  - D: Positive semi-definite matrix

- Polar decomposition is obtained from SVD
  $\mathrm{SVD}(\mathrm{M}) = W\Sigma V^T$ with $\mathrm{R} = \mathrm{W}\mathrm{V}^T$, $\mathrm{D} = \mathrm{V}\Sigma\mathrm{V}^T$

- Numerically, R can be computed using the following iterative scheme
  $R_0 = \mathrm{M}$, $R_{i+1} = 0.5\,(R_i + R_i^{-T})$

Handling affine transformation

Algorithm to interpolate between two general 4×4 matrix M_1, M_2

1. Extract translation p_1, p_2 from M_1, M_2.
2. Compute R_1, R_2 (3×3 rotation matrices), and D_1, D_2 (3×3 scaling/shearing matrices) from M_1, M_2
3. Interpolate linearly position and scaling/shearing $p(t) = (1-t)\, p_1 + t\, p_2$, $D(t) = (1-t)\, D_1 + t\, D_2$
4. Compute quaternions q_1, q_2 from R_1, R_2

   Note $M \to q$ with $q = \left(\frac{M_{zy} - M_{yz}}{2r}, \frac{M_{xz} - M_{zx}}{2r}, \frac{M_{yx} - M_{xy}}{2r}, \frac{r}{2} \right), r = \sqrt{1 + M_{xx} + M_{yy} + M_{zz}}$
5. Compute $q(t) = \mathrm{SLERP}(q_1, q_2, t)$
6. Convert back to matrix $q(t) \to R(t)$
7. Compute final matrix $M(t) = R(t)\, D(t)$ with translation $p(t)$.