

Affine Transformation

Affine/Linear Transformation

Most used deformations $p' = f(p)$

can be represented by a matrix $\rightarrow$ can be sent as uniform parameter to the shader.

- In standard 3D coordinates $p = (x, y, z)$

$$p' = L p + t$$

L : linear component (3×3 matrix)

t : translation

- In homogeneous coordinates $p = (x, y, z, 1)$

$$p' = A p$$

A : 4×4 matrix

$$A = \left(\begin{array}{ccc|c} L & t \\ \hline 0 & 1 \end{array} \right) = \left(\begin{array}{ccc|c} L_{xx} & L_{xy} & L_{xz} & t_x \\ L_{yx} & L_{yy} & L_{yz} & t_y \\ L_{zx} & L_{zy} & L_{zz} & t_z \\ \hline 0 & 0 & 0 & 1 \end{array} \right)$$

Properties:

- $\det(L)$: Change of volume when applying L to a shape

- $L^T L = 1 \Rightarrow$ Isometry

Affine/Linear Deformations

Translation

$$t(p) = (x + t_x, y + t_y, z + t_z)$$

$$T = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaling

$$s(p) = (s_x x, s_y y, s_z z)$$

$$S = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation

Several possible representations
(see later)

Note: Isometry

Shearing

$$sh_{xy}(p) = (x + \lambda y, y, z)$$

$$sh_{xz}(p) = (x + \lambda z, y, z)$$

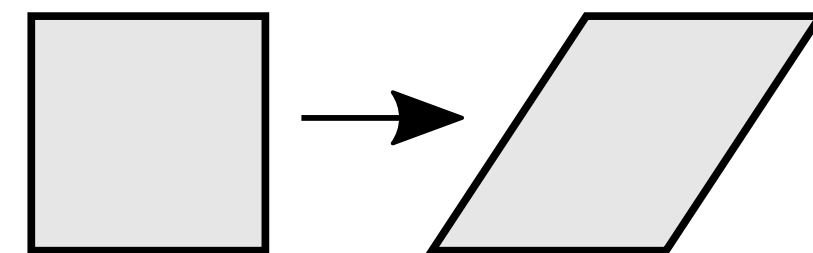
$$sh_{yx}(p) = (x, y + \lambda x, z)$$

...

$$Sh_{xy} = \begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad Sh_{xz} = \begin{pmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad Sh_{yx} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \dots$$

Usually avoided in Graphics

Note: $\det(Sh) = 1 \rightarrow$ constant volume (but no isometry)



Rotations

3D rotations have 3 dof, No unique representation

Matrix

$$R = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

$$R^T R = I$$

$$\det(R) = 1$$

- (+) Computationally convenient
- (-) Non-explicit dof, redundancies

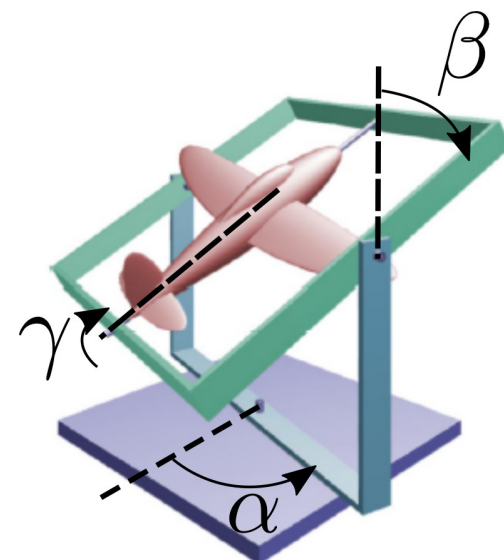
Euler Angles

3 angles: (α, β, γ)

Composition of rotation around basic axes

Not unique (x-y-z, y-z-x, x-y-x', x-z-x', ...)

- (+) Meaningfull parameters
- (-) Gimbal-lock

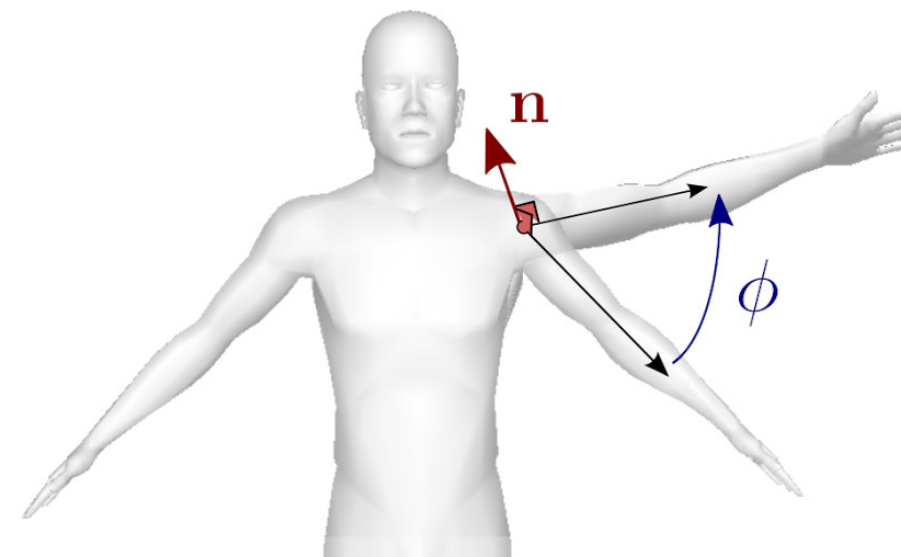


Axis angle

$(\mathbf{n}, \theta)$

(+) Meaningfull parameters

(-) No direct composition



Quaternion

$$q = (x, y, z, w) \\ = (\mathbf{n} \sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right))$$

- (+) Composition and interpolation
- (-) Less intuitive components

Cautions with transformations order

Take care, order of operation does matter !

Rotation r , Translation t : $r \circ t \neq t \circ r \Rightarrow M_1 = \mathbf{T} \mathbf{R} \neq \mathbf{R} \mathbf{T} = M_2$

Take care (2): transformation matrices applied to coordinates from right to left.

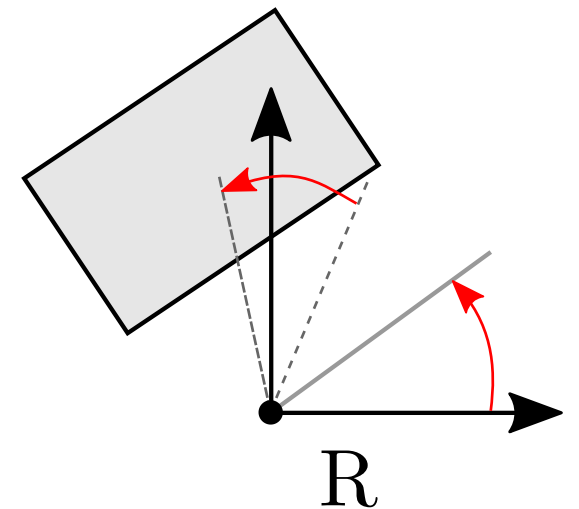
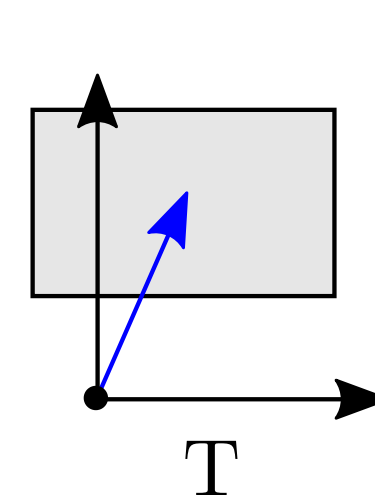
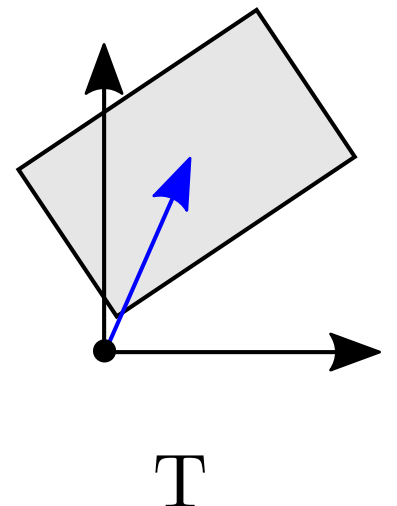
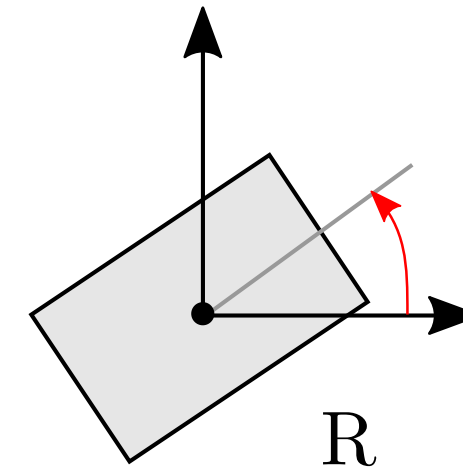
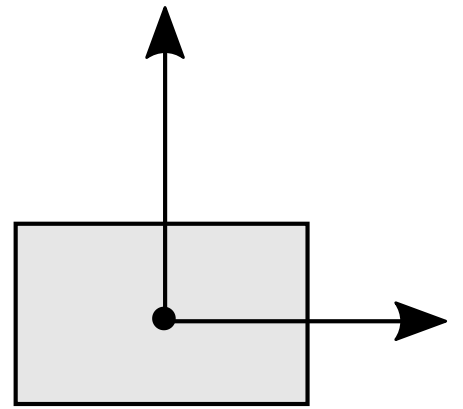
$$M_1 = \mathbf{T} \mathbf{R} = \left(\begin{array}{c|c} 1 & t \\ \hline 0 & 1 \end{array} \right) \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array} \right) = \left(\begin{array}{c|c} R & t \\ \hline 0 & 1 \end{array} \right)$$

First rotates, then translates

$$M_2 = \mathbf{R} \mathbf{T} = \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array} \right) \left(\begin{array}{c|c} 1 & t \\ \hline 0 & 1 \end{array} \right) = \left(\begin{array}{c|c} R & \mathbf{R} t \\ \hline 0 & 1 \end{array} \right)$$

First translates, then rotates

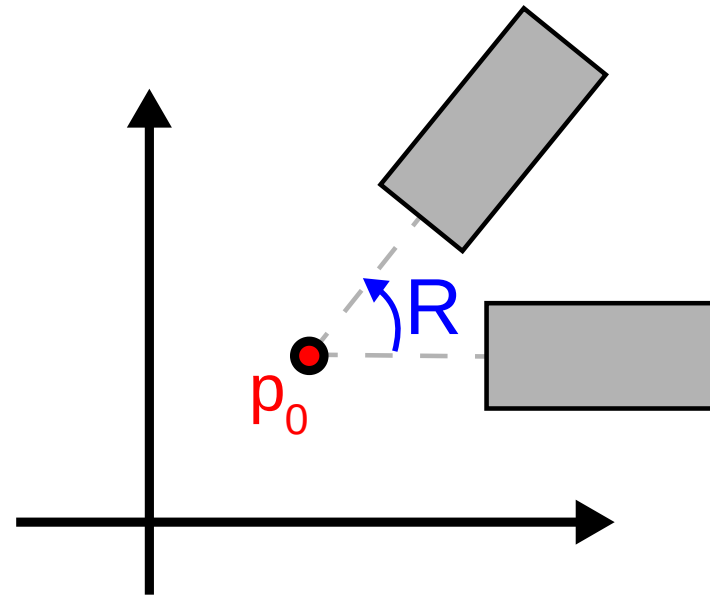
Rotation always happens around the origin.



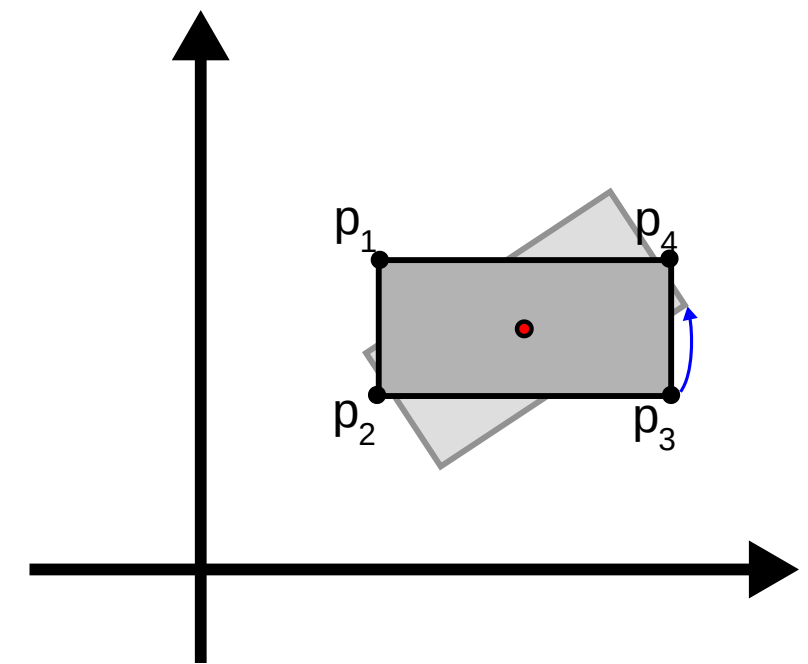
Affine Transformation Exercises

Q. Express the affine transform (as a 4×4 block matrix) corresponding to a rotation R applied around an arbitrary position p_0 in space.

> $M = \dots$



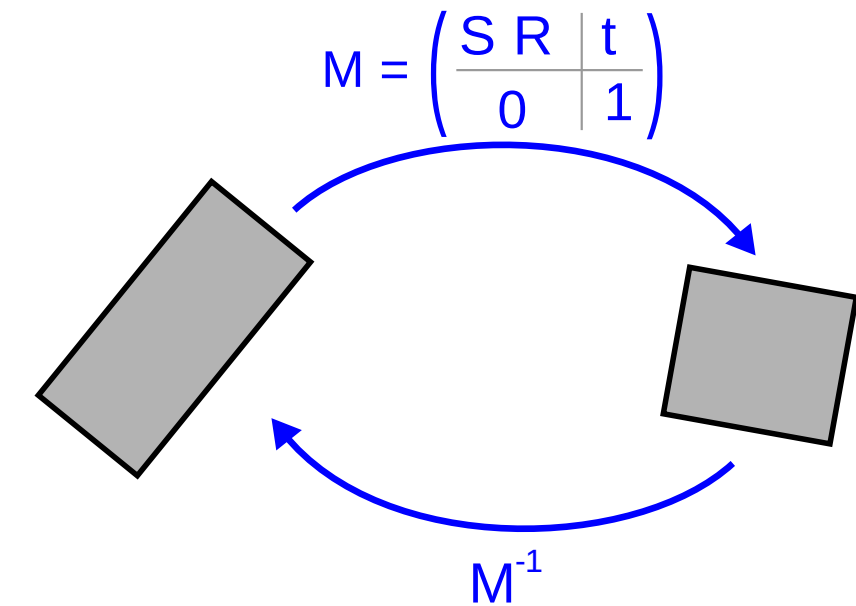
Q. Consider a shape defined by a triangular mesh with vertex positions $(p_i)_{i \in [1, N]}$.
> Express the affine transform allowing to rotate the shape around its barycenter.



Affine Transformation Exercises

Q. Consider the affine transform M parameterized by a scaling s , a rotation R , and a translation t .

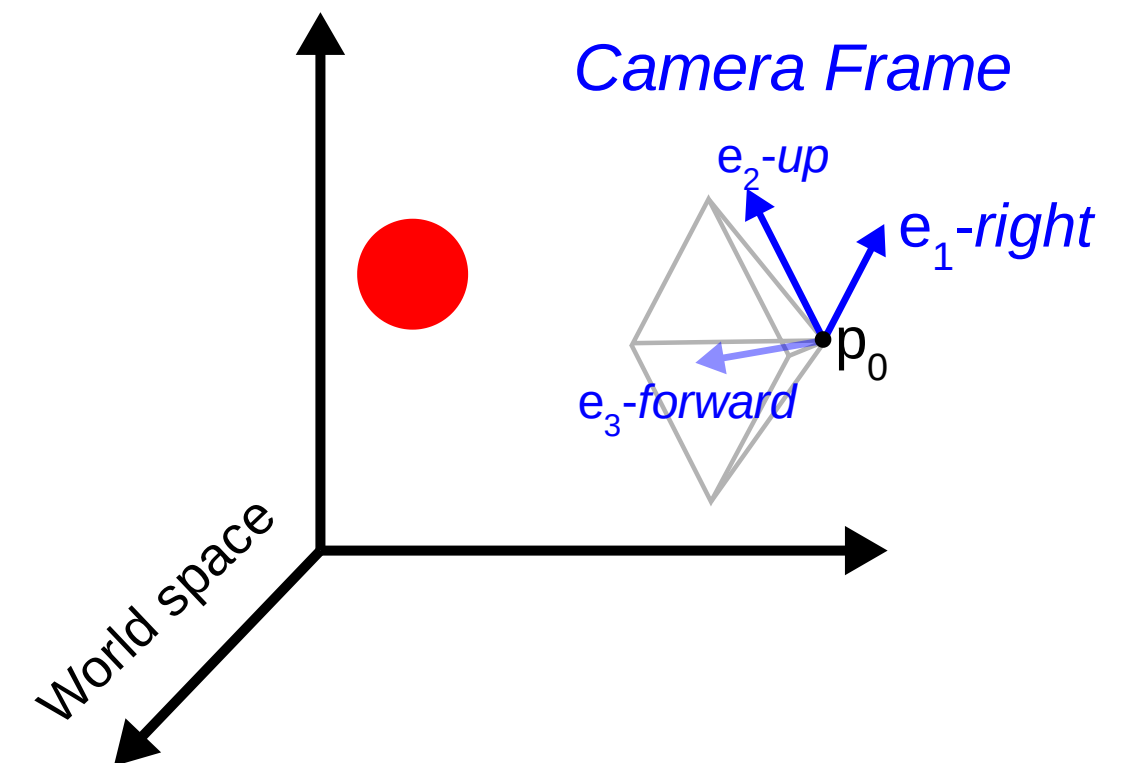
> Express the inverse matrix M^{-1} with respect to s , R and t



Q. Consider a camera described by its frame:

- a position p_0
- an orientation expressed by the three unit vectors (e_1, e_2, e_3) corresponding respectively to the right, up, and forward direction.

> Express the *view matrix* that should be applied in the shader, i.e. the matrix to apply to each 3D vertex position in order to model the effect of the camera position and orientation (3D world space $\rightarrow$ view space).



Rotations

Rotations

3D rotations have 3 dof, No unique representation

Matrix

$$R = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

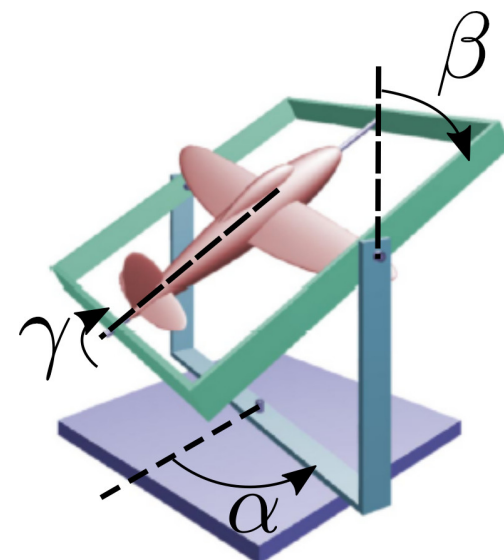
$$R^T R = I$$

$$\det(R) = 1$$

Euler Angles

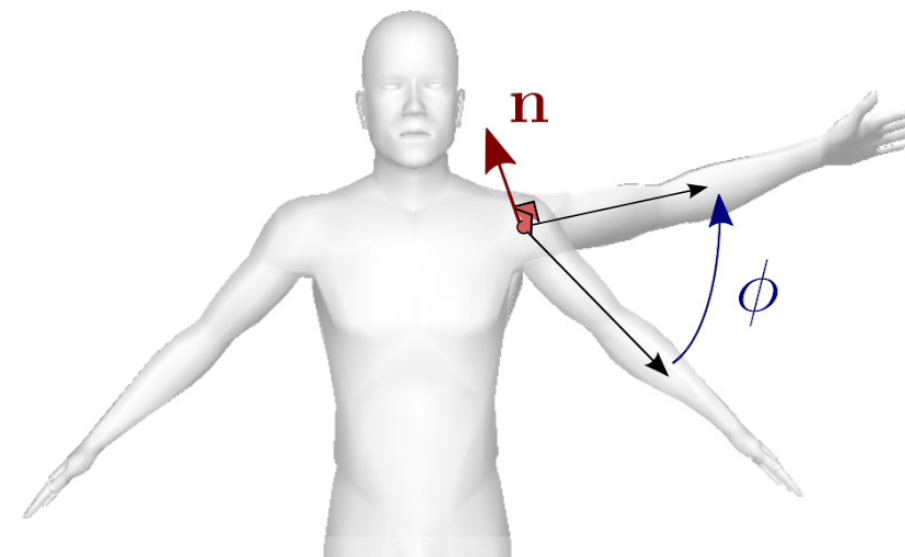
3 angles: (α, β, γ)

Composition of rotation
around basic axes



Axis angle

$(\mathbf{n}, \theta)$



Quaternion

$$q = (x, y, z, w) \\ = \left(\mathbf{n} \sin \left(\frac{\theta}{2} \right), \cos \left(\frac{\theta}{2} \right) \right)$$

Rotation: Focus Euler angles

Three consecutive rotations along fixed axis along (x, y, z) coordinates.

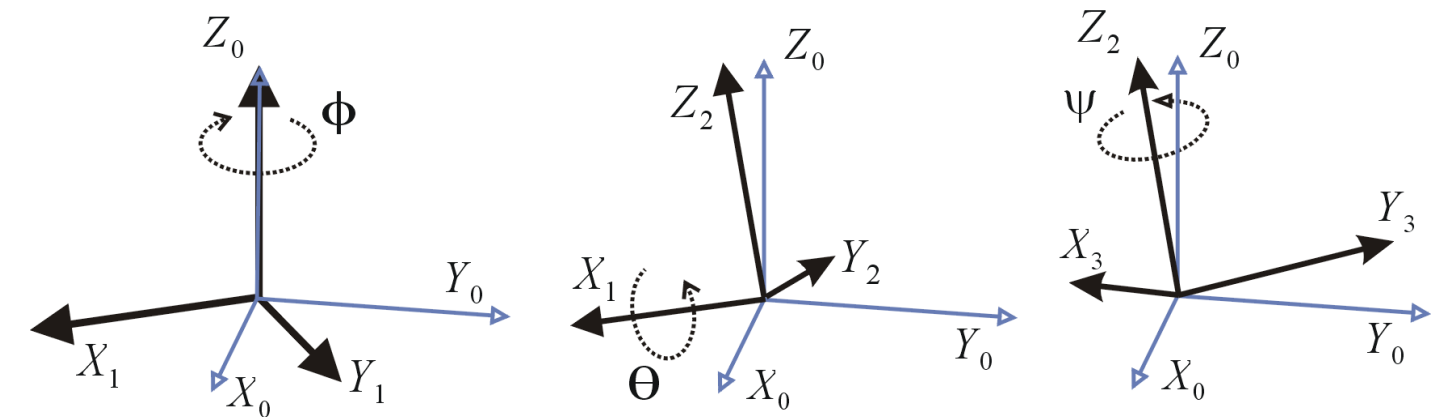
Can use basic rotation matrices composition

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad R_y = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix} \quad R_z = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiple Euler angles conventions

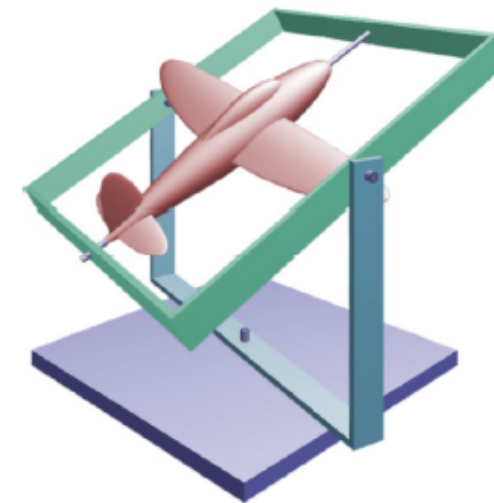
- *Proper Euler* : z - x - z' , x - y - x' , y - z - y' , ...
- *Tait-Bryan* : x - y - z , z - y - x , x - z - y , ...

Take care when exporting/importing/parsing between Softwares



Pro

- Combination of rotation around known axis
- Comprehensive parameters (3 dof)
- Animators can interact with angular curves
- *Widely used in robotics*



Rotation: Focus Euler angles

Limitations of Euler Angle

- *Gimbal Lock* when composing b/w some rotations

Loose one degree of freedom in specific configuration

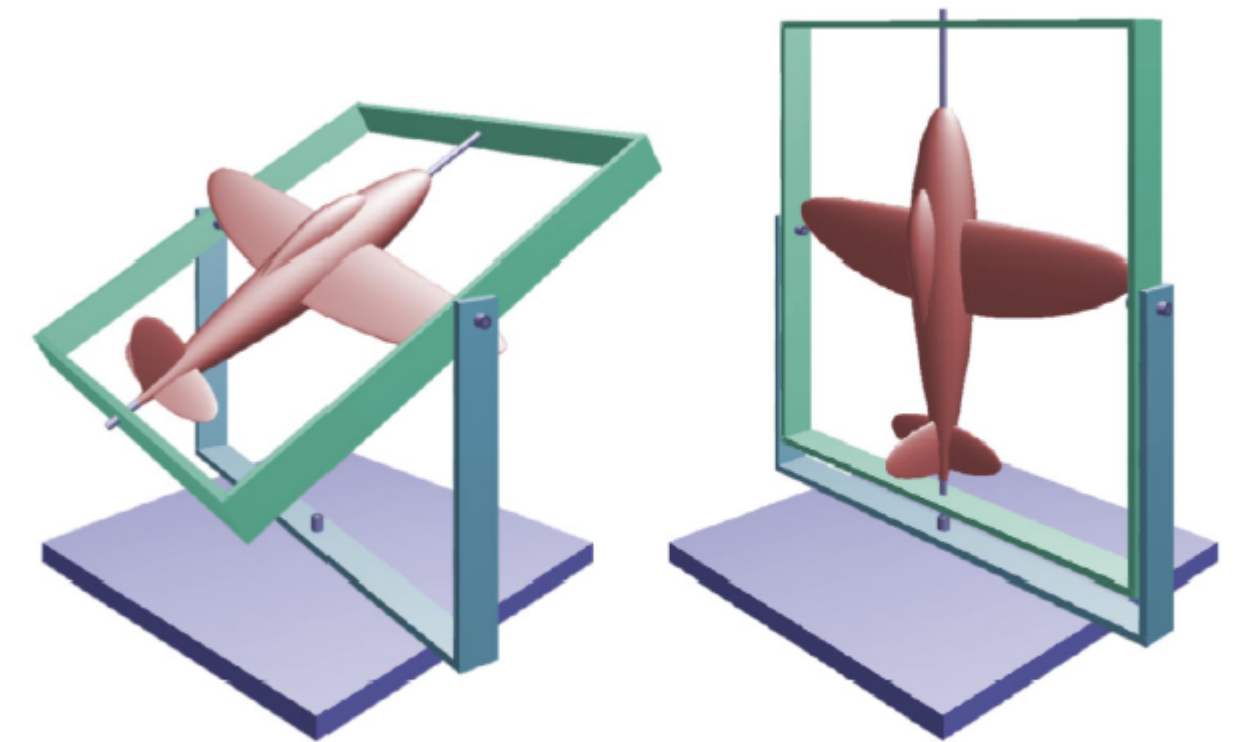
$$\text{ex. } R_x(\alpha) R_y(\pi/2) R_z(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha) \cos(\gamma) + \cos(\alpha) \sin(\gamma) & -\sin(\alpha) \sin(\gamma) + \cos(\alpha) \cos(\gamma) & 0 \\ -\cos(\alpha) \cos(\gamma) + \sin(\alpha) \sin(\gamma) & \cos(\alpha) \sin(\gamma) + \sin(\alpha) \cos(\gamma) & 0 \end{pmatrix}$$

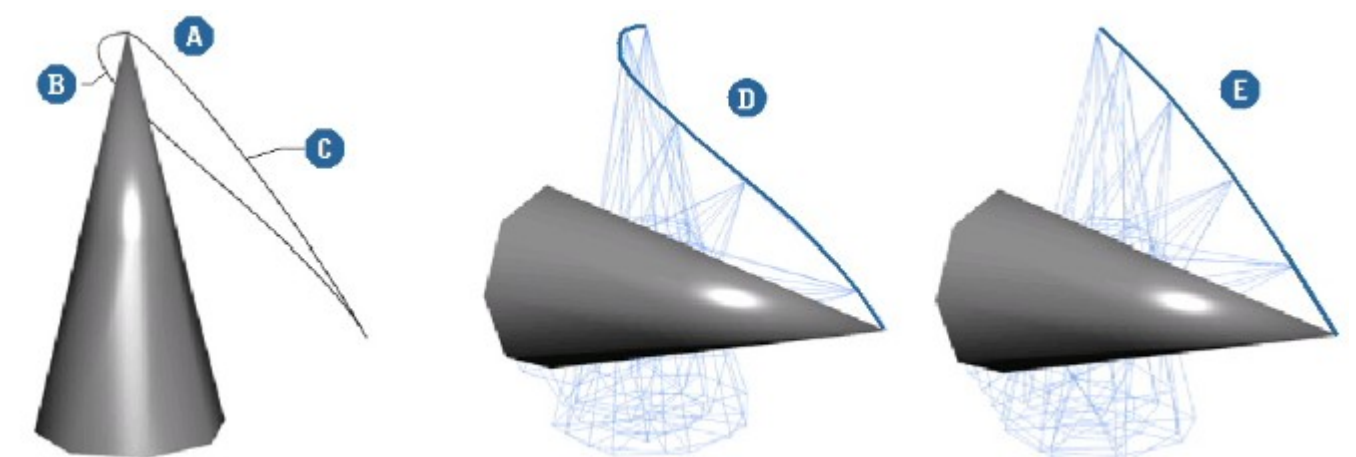
$$= \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ -\cos(\alpha + \gamma) & \sin(\alpha + \gamma) & 0 \end{pmatrix}$$

Expect 2-dof, but get 1-dof

- Interpolation of 3 angles possible but not necessarily with the simplest trajectory.



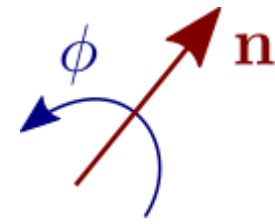
<http://www.fho-emden.de/~hoffmann/gimbal09082002.pdf>



Rotation: Focus Axis Angle

Any 3D rotation can be represented by

- A unit axis n
- An angle θ

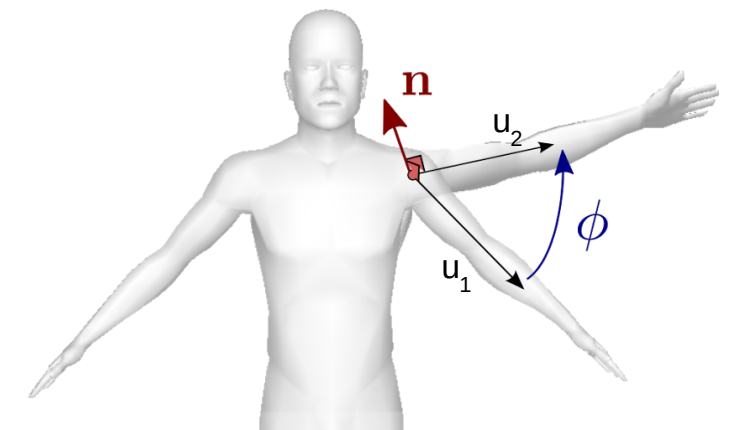
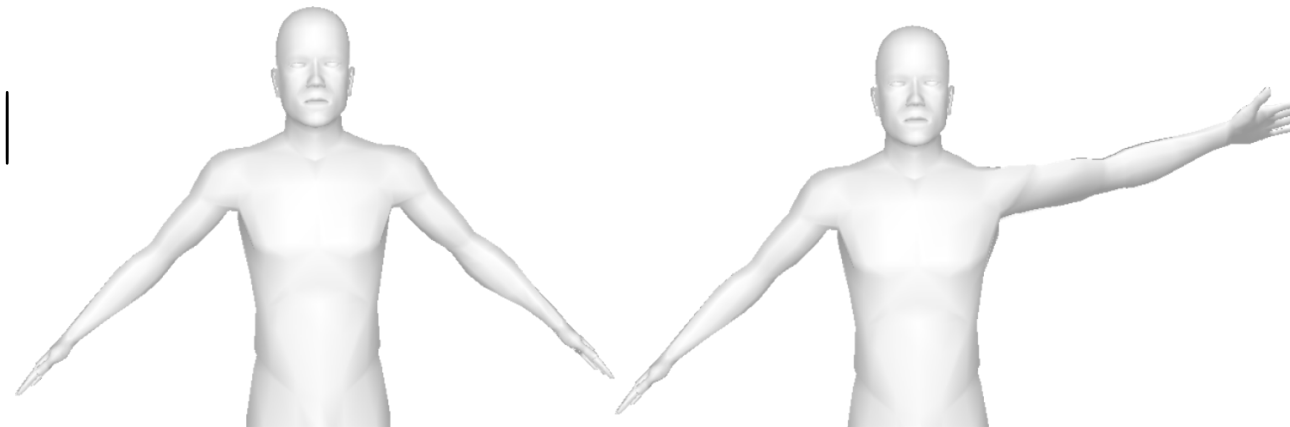


Pro.

- Concise representation: 3 DOF
- Meaningful parameters
- Describes well the rotation between two vectors

Ex. Rotation between u_1 and u_2

- Axis of rotation $n = (u_1 \times u_2) / \|u_1 \times u_2\|$
- Angle of rotation $\theta = \arccos(u_1 \cdot u_2)$
- Twist around u_2 can be arbitrarily chosen



Rotation: Focus Axis Angle - Rodrigues

Applying a rotation (n, θ) to a vector v

$$v = v_{\parallel} + v_{\perp}$$

$$v' = v'_{\parallel} + v'_{\perp}$$

$$\Rightarrow v' = v_{\parallel} + (\cos(\theta) v_{\perp} + \sin(\theta) n \times v_{\perp})$$

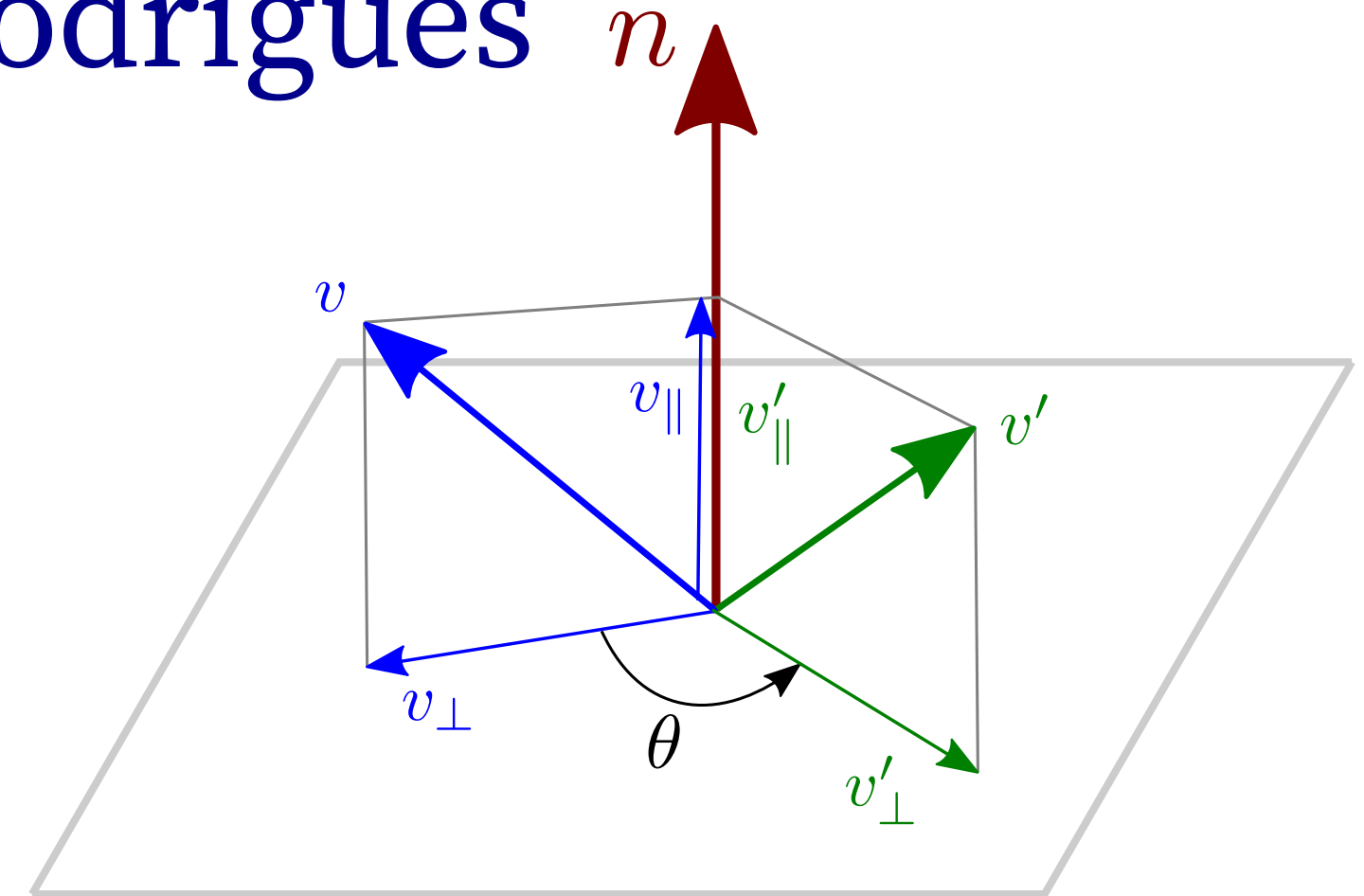
$$v_{\parallel} = (v \cdot n) n$$

$$v_{\perp} = v - (v \cdot n) n$$

$$v' = (v \cdot n) n + \cos(\theta)(v - (v \cdot n) n) + \sin(\theta) n \times (v - (v \cdot n) n)$$

$$\Rightarrow v' = \cos(\theta) v + \sin(\theta) n \times v + (1 - \cos(\theta)) (v \cdot n) n$$

Rodrigues' rotation Formula



Rotation: Focus Axis Angle as Matrix

$$v' = \cos(\theta) v + \sin(\theta) n \times v + (1 - \cos(\theta)) (v \cdot n) n = R(n, \theta) v$$

$$v' = \cos(\theta) v + \sin(\theta) \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \times v + (1 - \cos(\theta)) \begin{pmatrix} n_x & n_y & n_z \end{pmatrix} v \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

$$v' = \cos(\theta) v + \sin(\theta) \underbrace{\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}}_K v + (1 - \cos(\theta)) \underbrace{\begin{pmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{pmatrix}}_{K^2} v$$

$$v' = \begin{pmatrix} \cos(\theta) + n_x^2(1 - \cos(\theta)) & n_x n_y(1 - \cos(\theta)) - n_z \sin(\theta) & n_x n_z(1 - \cos(\theta)) + n_y \sin(\theta) \\ n_x n_y(1 - \cos(\theta)) + n_z \sin(\theta) & \cos(\theta) + n_y^2(1 - \cos(\theta)) & n_y n_z(1 - \cos(\theta)) - n_x \sin(\theta) \\ n_x n_z(1 - \cos(\theta)) - n_y \sin(\theta) & n_y n_z(1 - \cos(\theta)) + n_x \sin(\theta) & \cos(\theta) + n_z^2(1 - \cos(\theta)) \end{pmatrix} v$$

Rotation: Focus Axis Angle - Summary

Given a rotation (n, θ) - Corresponding rotation matrix

$$R(n, \theta) = I + \sin(\theta) K + (1 - \cos(\theta)) K^2$$

$$R(n, \theta) = \begin{pmatrix} \cos(\theta) + n_x^2(1 - \cos(\theta)) & n_x n_y(1 - \cos(\theta)) - n_z \sin(\theta) & n_x n_z(1 - \cos(\theta)) + n_y \sin(\theta) \\ n_x n_y(1 - \cos(\theta)) + n_z \sin(\theta) & \cos(\theta) + n_y^2(1 - \cos(\theta)) & n_y n_z(1 - \cos(\theta)) - n_x \sin(\theta) \\ n_x n_z(1 - \cos(\theta)) - n_y \sin(\theta) & n_y n_z(1 - \cos(\theta)) + n_x \sin(\theta) & \cos(\theta) + n_z^2(1 - \cos(\theta)) \end{pmatrix}$$

Pro

- Concise and general representation
- Expressive parameters in 3D space (axe, angles)
- Efficient rotation and correspondance with matrix

Cons

- No simple composition expression between two rotations.
- No direct interpolation

(well handled by quaternion representation)

Rotation: Focus on Quaternions

Quaternions: generalization of complex numbers.

$$q = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} + w \quad w \text{ real part, } (x, y, z) \text{ imaginary (or pure quaternion) part.}$$

We write in short $q = (x, y, z, w)$

(don't confound with 4D vectors in homogeneous coordinates)

Properties of imaginary basis vectors

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

$$\mathbf{ijk} = -1$$

- Provides the algebraic properties

Basics operations on quaternions

- Conjugated quaternion $q^* = (-x, -y, -z, w)$

- Quaternion norm $\|q\| = \sqrt{q q^*} = \sqrt{x^2 + y^2 + z^2 + w^2}$. Unit quaternion satisfies $\|q\| = 1$.

- Quaternion product

$$q_1 q_2 = (x_1 \mathbf{i} + y_1 \mathbf{j} + z_1 \mathbf{k} + w_1) (x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k} + w_2) = \dots$$

$$q_1 q_2 = \begin{pmatrix} x_1 w_2 + w_1 x_2 + y_1 z_2 - z_1 y_2 \\ y_1 w_2 + w_1 y_2 + z_1 x_2 - x_1 z_2 \\ z_1 w_2 + w_1 z_2 + x_1 y_2 - y_1 x_2 \\ w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 \end{pmatrix}$$

Note: Quaternion product is

- associative: $(q_1 q_2) q_3 = q_1 (q_2 q_3) = q_1 q_2 q_3$
- **non-commutative**: $q_1 q_2 \neq q_2 q_1$

Sometimes interesting to separate *real part* w from *pure quaternion* part $s = (x, y, z)$.

- Shorthand vector form $q = (s, w)$

- Quaternion product in vector form $q_1 q_2 = (s_1 w_2 + s_2 w_1 + s_1 \times s_2, w_1 w_2 - s_1 \cdot s_2)$

Relation between quaternion and rotation

Consider

- a quaternion $q = (s, w)$ of unit norm $\|q\| = 1$.
- a vector $v = (v_x, v_y, v_z)$ assimilated to the pure quaternion $q_v = (v, 0) = (v_x, v_y, v_z, 0)$.

Then

- $q_{v'} = \mathcal{R}_q(v) = q q_v q^*$ is a pure quaternion $q_{v'} = (v'_x, v'_y, v'_z, 0)$
- And $v' = (v'_x, v'_y, v'_z)$ is the rotation of vector v around the axis $n = s/\|s\|$, with angle $2 \arccos(w)$.

Demonstration

$$\mathcal{R}_q(v) = (s, w) (v, 0) (-s, w) = \dots = ((w^2 - s^2) v + 2(s \cdot v) s + 2w (s \times v), 0)$$

As $\|q\| = 1$, we can write $q = (s, w) = (n \sin(\phi), \cos(\phi))$, where $\|n\| = 1$

$$\text{Then } \mathcal{R}_q(v) = \underbrace{((\cos^2(\phi) - \sin^2(\phi)) v)}_{\cos(2\phi)} + \underbrace{2 \sin^2(\phi)(n \cdot v) n}_{1 - \cos(2\phi)} + \underbrace{2 \cos(\phi) \sin(\phi) n \times v}_{\sin(2\phi)}, 0)$$

$\Rightarrow$ Rodrigues formula for axis n and angle 2ϕ .

The unit quaternion $q = (n \sin(\theta/2), \cos(\theta/2))$ represents the rotation of angle θ around the axis n .

Composition of rotations

Consider two rotation (R_1, R_2) associated to their unit quaternions (q_1, q_2) .

The product $q_1 q_2$ represents the composition $R_1 \circ R_2$.

Demonstration

We show that $\mathcal{R}_{q_1 q_2}(v) = \mathcal{R}_{q_1} \circ \mathcal{R}_{q_2}(v)$.

$$\mathcal{R}_{q_1 q_2}(v) = (q_1 q_2) v (q_1 q_2)^*$$

$$\mathcal{R}_{q_1 q_2}(v) = (q_1 q_2) v (q_2^* q_1^*), \text{ as } (q_1 q_2)^* = q_2^* q_1^*$$

$$\mathcal{R}_{q_1 q_2}(v) = q_1 (q_2 v q_2^*) q_1^*$$

$$\mathcal{R}_{q_1 q_2}(v) = q_1 \mathcal{R}_{q_2}(v) q_1^* = \mathcal{R}_{q_1} \circ \mathcal{R}_{q_2}(v)$$

Correspondance quaternion to rotation matrix

The unit quaternion $q = (x, y, z, w)$ represents the rotation given by the matrix

$$R = \begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{pmatrix}$$

Demonstration

$$v' = \mathcal{R}_q(v) = q q_v q^\star = ((w^2 - s^2) v + 2(s \cdot v) s + 2w (s \times v), 0) \quad \text{with } s = (x, y, z)$$

$$v' = (w^2 - x^2 - y^2 - z^2)v + 2 \begin{pmatrix} x & y & z \end{pmatrix} v \begin{pmatrix} x & y & z \end{pmatrix}^T + 2w \begin{pmatrix} x & y & z \end{pmatrix} \times v$$

$$v' = \left((w^2 - x^2 - y^2 - z^2) \mathbf{I} + 2 \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} + 2w \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \right) v$$

$$v' = R v, \text{ with } x^2 + y^2 + z^2 + w^2 = 1$$

Summary - Correspondance quaternion / matrix-vector

Representation and Operations

	3D space	Quaternion space
<i>Vector</i>	$v = (v_x, v_y, v_z)$	$q_v = (v, 0) = (v_x, v_y, v_z, 0)$
<i>Rotation</i>	R (3×3 matrix)	$q = (x, y, z, w), \ q\ = 1$
<i>Apply rotation to vector</i>	Rv	$q q_v q^\star$
<i>Rotation composition</i>	$R_1 R_2$	$q_1 q_2$

Rotation to Quaternion

Rotation of axis n and angle $\theta \Rightarrow q = (n \sin(\theta/2), \cos(\theta/2))$

Quaternion to Rotation

$$\text{Unit quaternion } q = (x, y, z, w) \Rightarrow R = \begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{pmatrix}$$

Interpolation

Rotations

Interpolating rotation

Problem: Given 2 rotations r_1, r_2

- Find rotation $r(t)$ st $r(0) = r_1, r(1) = r_2$, and varies smoothly along t

Matrix representation

Componentwise interpolation not adapted

ex. $t R_1 + (1 - t) R_2$ is not a rotation: introduce scaling/shearing

The correct formulation to interpolate on the manifold would be $R_1 \exp(t \log(R_1^T R_2))$

But matrix exponential is complex to compute

Euler angle

Can interpolate the 3 angles separately

(+) Leads to a rotation

(-) Doesn't necessarily follows simplest trajectory

Axis-Angle

No trivial interpolation scheme with different axes

Quaternion

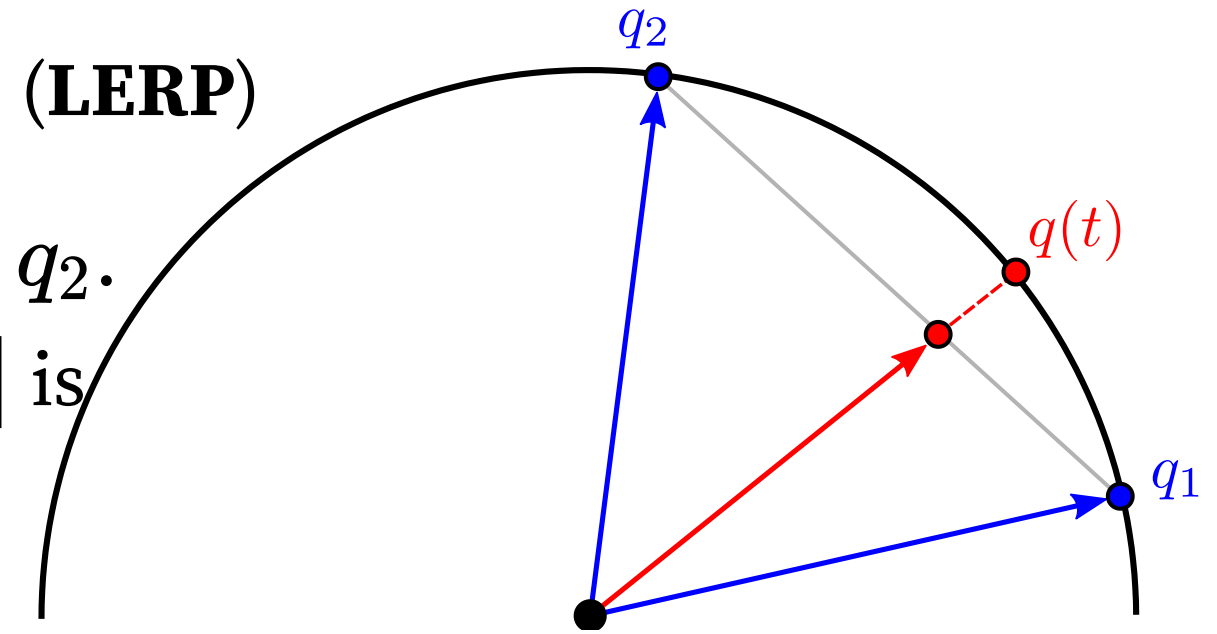
Well adapted (see later)

Interpolation of Quaternion - LERP

Linear interpolation (with normalization) in quaternion space (**LERP**)

- Consider two rotations with corresponding quaternion q_1, q_2 .
- The *linearly interpolated* quaternion at parameter $t \in [0, 1]$ is

$$q(t) = \frac{(1 - t) q_1 + t q_2}{\|(1 - t) q_1 + t q_2\|}$$



- (+) Follows a shortest path (great circle on 4D sphere)
- (+) Can be generalized to more general parametric curves (Splines, etc)
- (-) Angular speed of orientation is not-constant
ex. extreme case of two opposite quaternions

Interpolation of Quaternion - SLERP

Spherical Linear interpolation (**SLERP**)

- Consider two unit vectors (arbitrary dimensions) v_1, v_2 .
- The *spherical linear interpolation* at parameter $t \in [0, 1]$ is

$$v(t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)} v_1 + \frac{\sin(t\Omega)}{\sin(\Omega)} v_2 \quad \text{with } \cos(\Omega) = v_1 \cdot v_2$$

Demonstration

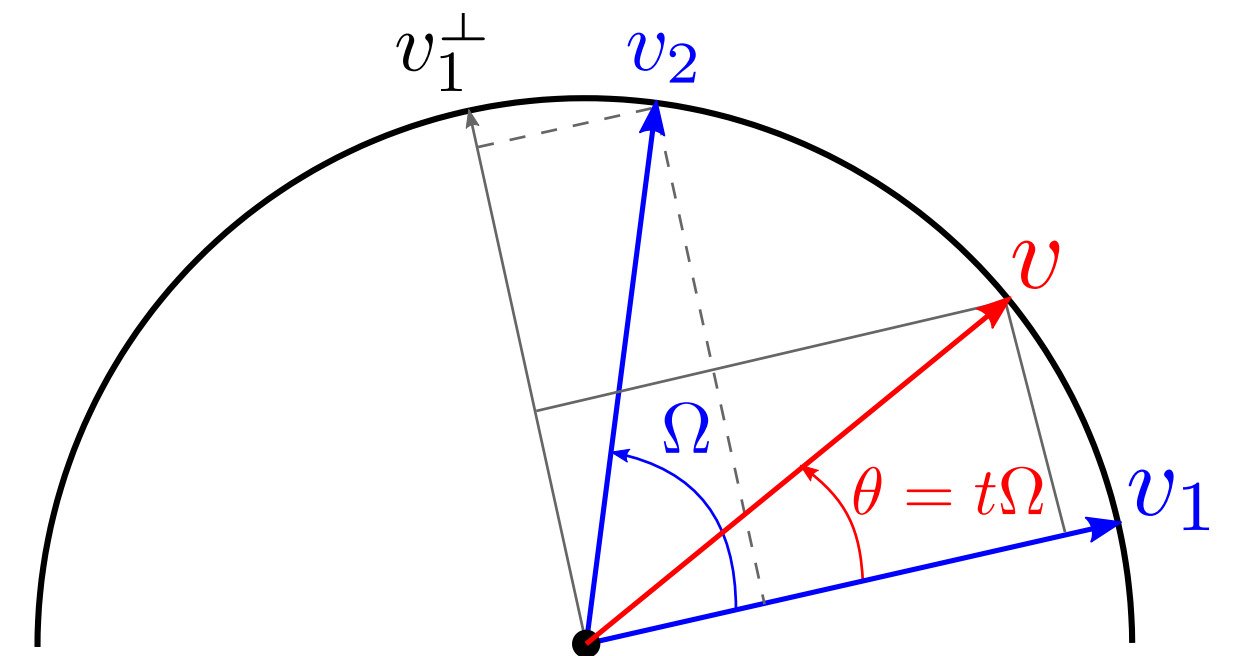
Consider

- Two unit vectors (arbitrary dimensions) v_1, v_2 .
- Ω : angle b/w v_1 and v_2
- The interpolated vector v at angle $\theta = \Omega t, t \in [0, 1]$.

$$v = v_1 \cos(\theta) + v_1^\perp \sin(\theta), \text{ and } v_1^\perp = \frac{v_2 - \cos(\Omega) v_1}{\sin(\Omega)}$$

$$\Rightarrow v = \left(\frac{\sin(\Omega) \cos(\theta) - \cos(\Omega) \sin(\theta)}{\sin(\Omega)} \right) v_1 + \frac{\sin(\theta)}{\sin(\Omega)} v_2$$

$$\Rightarrow v = \frac{\sin(\Omega - \theta)}{\sin(\Omega)} v_1 + \frac{\sin(\theta)}{\sin(\Omega)} v_2$$

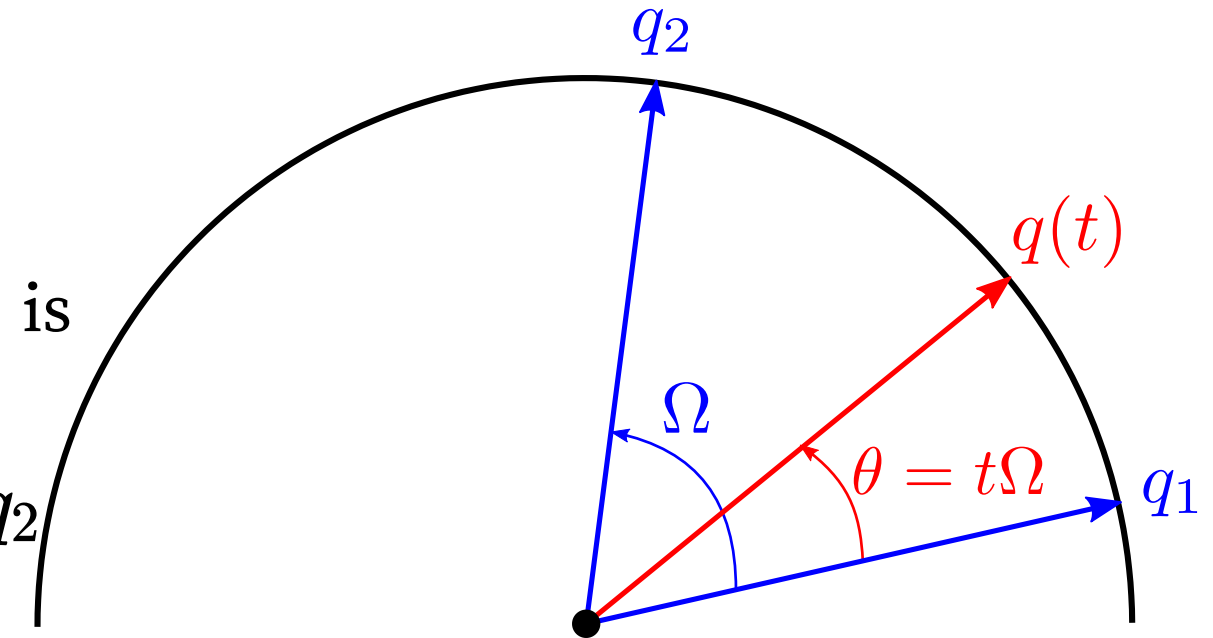


Interpolation of Quaternion - SLERP

Spherical Linear interpolation (**SLERP**)

- Consider two rotations with corresponding quaternion q_1, q_2 .
- The *spherical linear interpolated* quaternion at parameter $t \in [0, 1]$ is

$$q(t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)} q_1 + \frac{\sin(t\Omega)}{\sin(\Omega)} q_2 \quad \text{with } \cos(\Omega) = q_1 \cdot q_2$$



(+) Follows a shortest path (great circle on 4D sphere)

(+) Constant angular speed

(-) Cannot be generalized to more general curves.

Cannot interpolate between more than two quaternions

Care with quaternion negation

$+q$ and $-q$ correspond to the same rotation ($n \rightarrow -n, \theta \rightarrow 2\pi - \theta$)

*Warning $-q$ **does not** corresponds to the rotation matrix $-R$.*

But to a **different path** when interpolated in the 4D quaternion space.

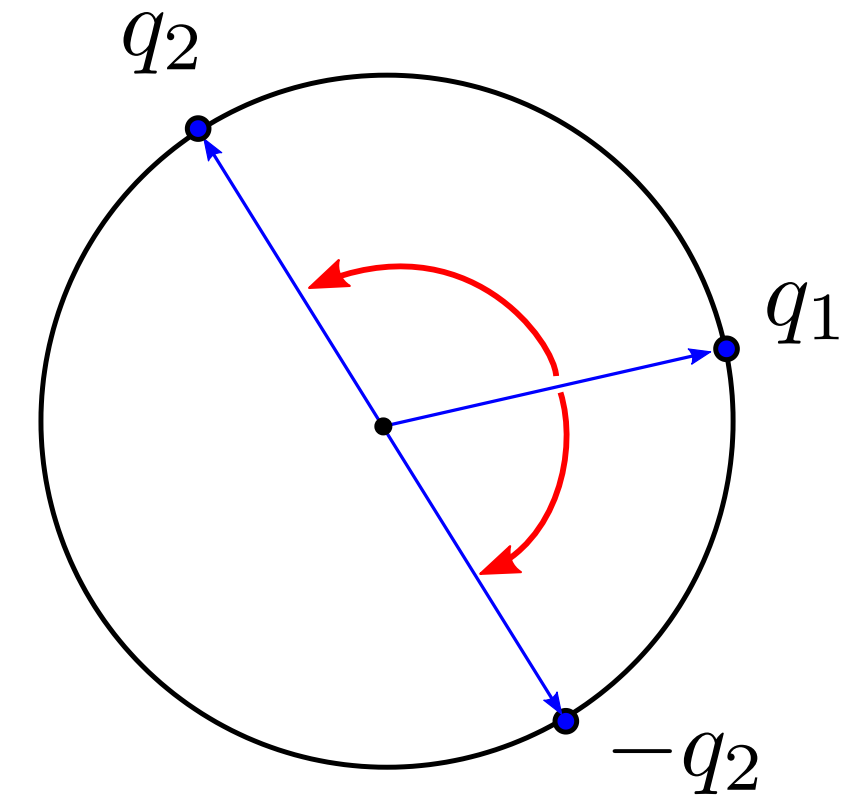
$\Rightarrow$ path $q_1 \rightarrow -q_2$ is shorter than $q_1 \rightarrow q_2$ when $q_1 \cdot q_2 < 0$.

In practice we check for the shorter path before applying SLERP.

Algorithm

```
if( dot(q1,q2)<0 )  
    q2 = -q2
```

```
q(t) = SLERP(q1,q2,t)
```



Interpolation

Affine Transforms

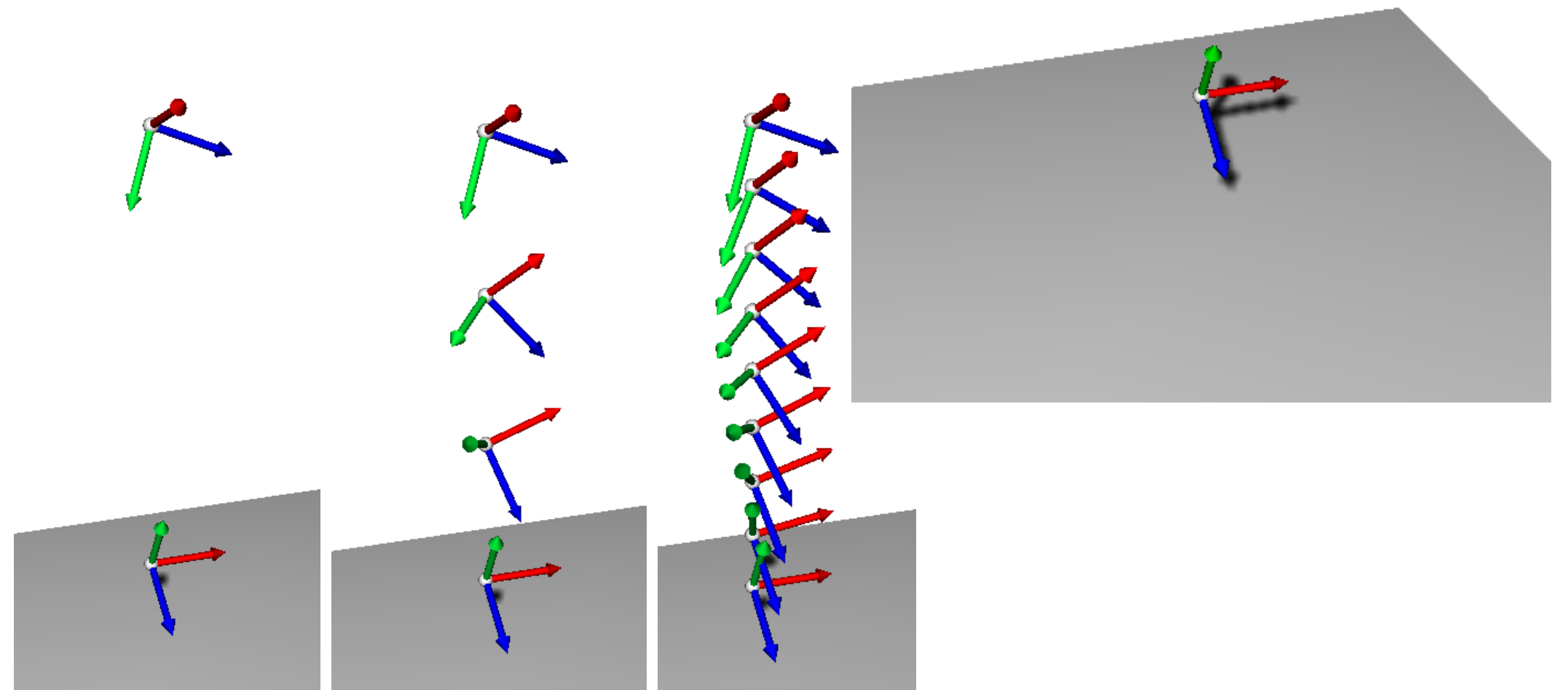
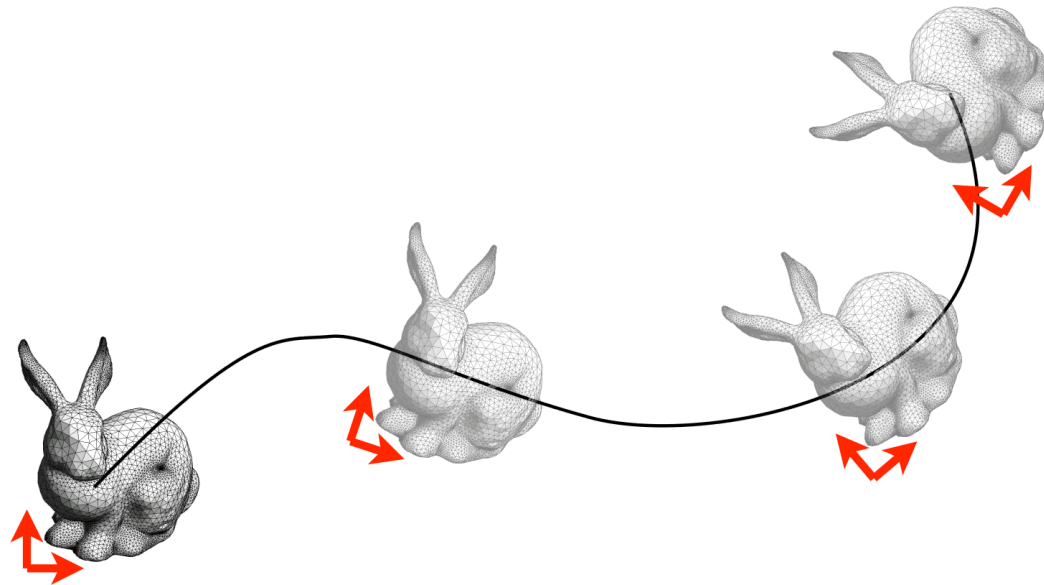
Interpolating rigid motion

3D objects have orientation r and position p .

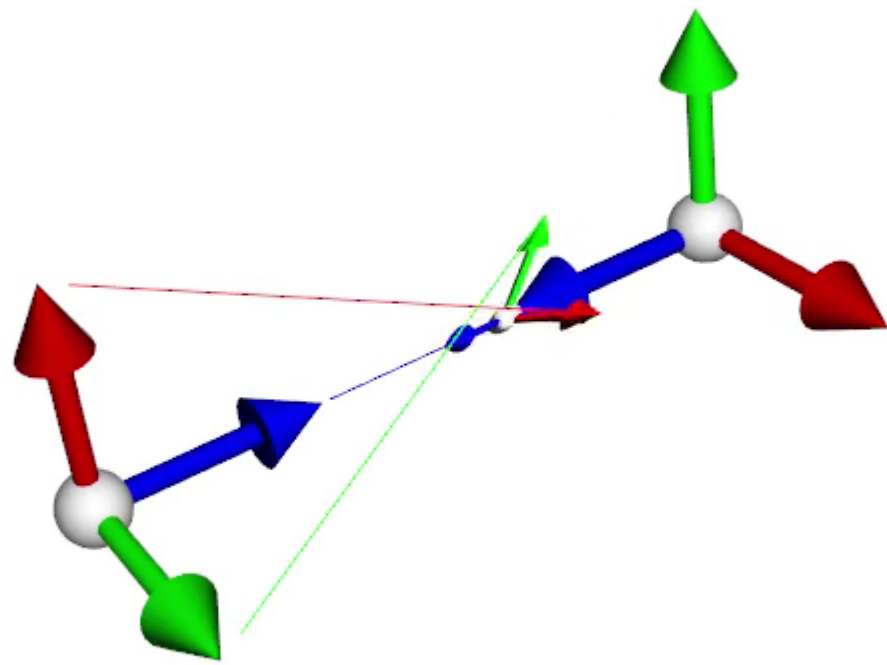
⇒ Need to interpolate both, usually handled separately.

Given two key-positions (p_1, r_1) and (p_2, r_2) *in-betweens* computed at time $t \in [0, 1]$ as

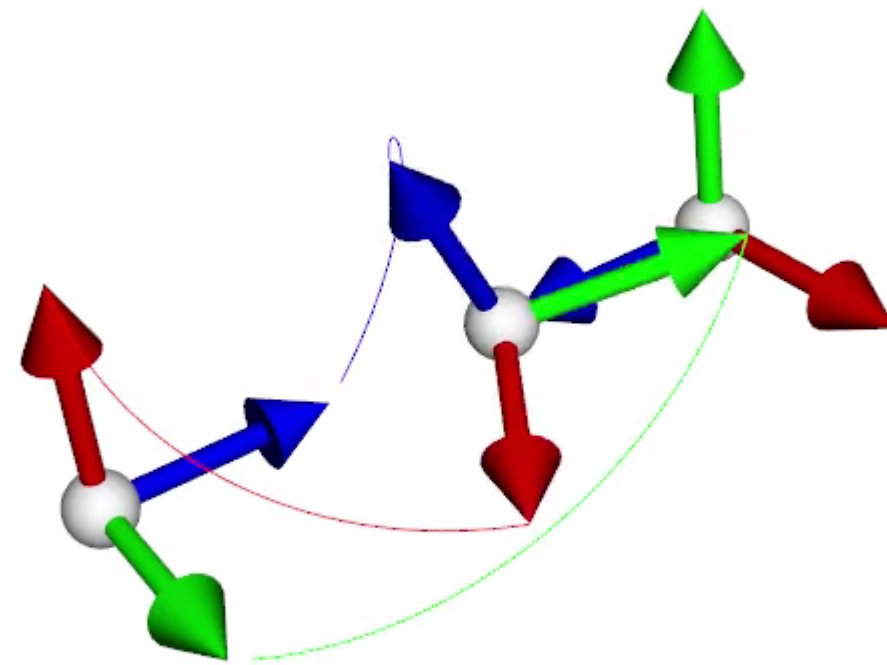
- Linear interpolation of positions $p(t) = (1 - t)p_1 + tp_2$
- Interpolate rotation with SLERP on quaternions
 - Convert $(r_1, r_2) \rightarrow (q_1, q_2)$
 - Compute $q(t) = \text{SLERP}(q_1, q_2, t)$
 - Convert back $q(t) \rightarrow r(t)$



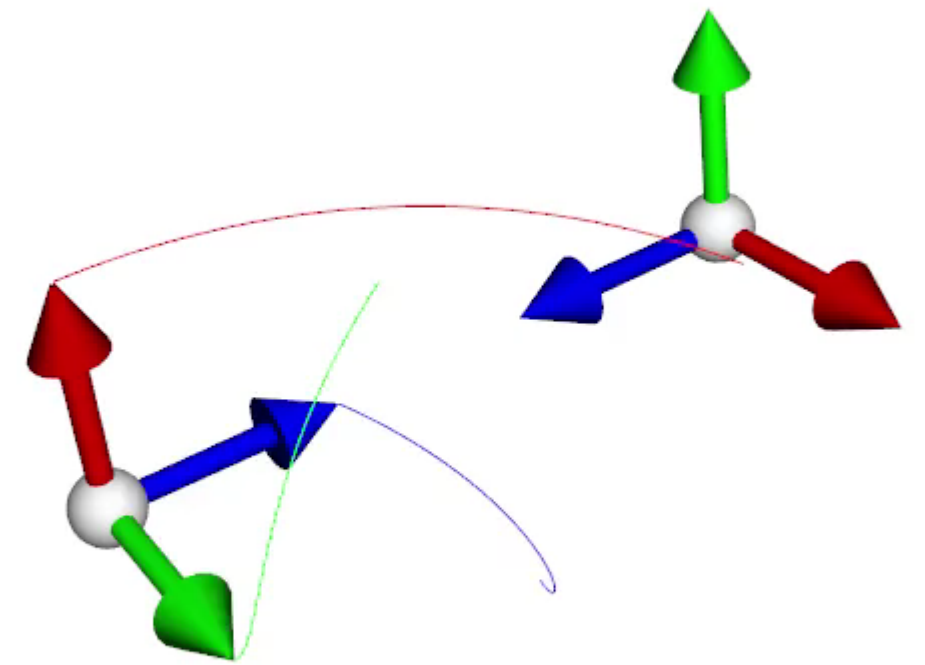
Interpolating rigid motion - Comparison



Matrix interpolation



Euler angle interpolation



Quaternion interpolation

Handling affine transformation : Polar Decomposition

Key-pose can be given as matrix M containing rotation, but also scaling and shearing.

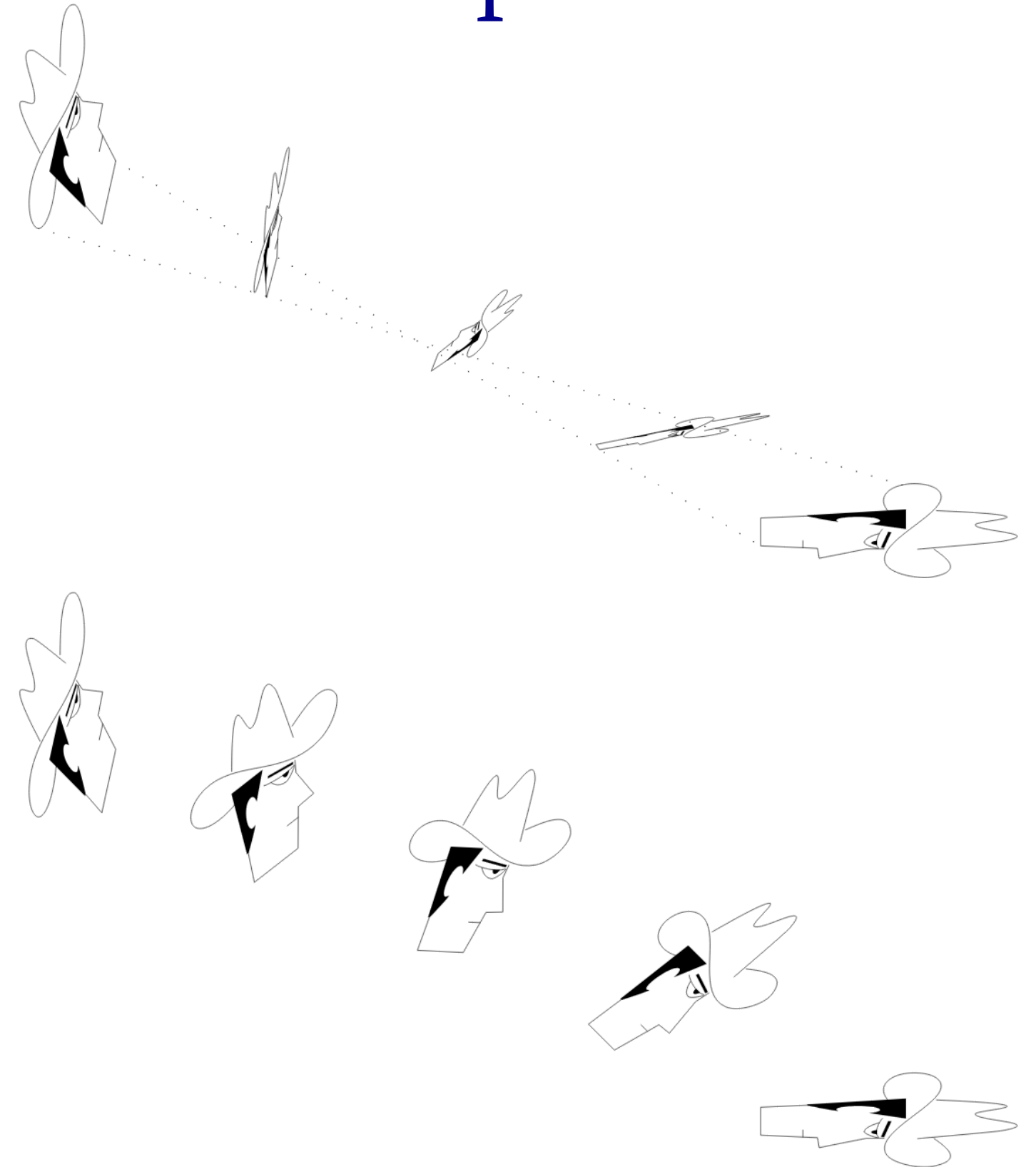
⇒ Cannot convert M to quaternion (not a rotation).

⇒ Separate *rotation* component from shearing and scaling component using **polar decomposition**.

[K. Shoemake and T. Duff, Matrix Animation and Polar Decomposition. Graphics Interface 92.]

- Polar decomposition: $M = R D$
 - R : Rotation matrix
 - D : Positive semi-definite matrix
- Polar decomposition is obtained from SVD
$$SVD(M) = W \Sigma V^T$$
 with
$$R = W V^T$$
$$, D = V \Sigma V^T$$
- Numerically, R can be computed using the following iterative scheme

$$R_0 = M, R_{i+1} = 0.5 (R_i + (R_i^{-1})^T)$$



Handling affine transformation

Algorithm to interpolate between two general 4×4 matrices M_1, M_2

1. Extract translation p_1, p_2 from M_1, M_2 .
2. Compute R_1, R_2 (3×3 rotation matrices), and D_1, D_2 (3×3 scaling/shearing matrices) from M_1, M_2
3. Interpolate linearly position and scaling/shearing $p(t) = (1 - t) p_1 + t p_2, D(t) = (1 - t) D_1 + t D_2$
4. Compute quaternions q_1, q_2 from R_1, R_2

Note $M \rightarrow q$ with $q = \left(\frac{M_{zy} - M_{yz}}{2r}, \frac{M_{xz} - M_{zx}}{2r}, \frac{M_{yx} - M_{xy}}{2r}, \frac{r}{2} \right), r = \sqrt{1 + M_{xx} + M_{yy} + M_{zz}}$

5. Compute $q(t) = \text{SLERP}(q_1, q_2, t)$
6. Convert back to matrix $q(t) \rightarrow R(t)$
7. Compute final matrix $M(t) = R(t) D(t)$ with translation $p(t)$.