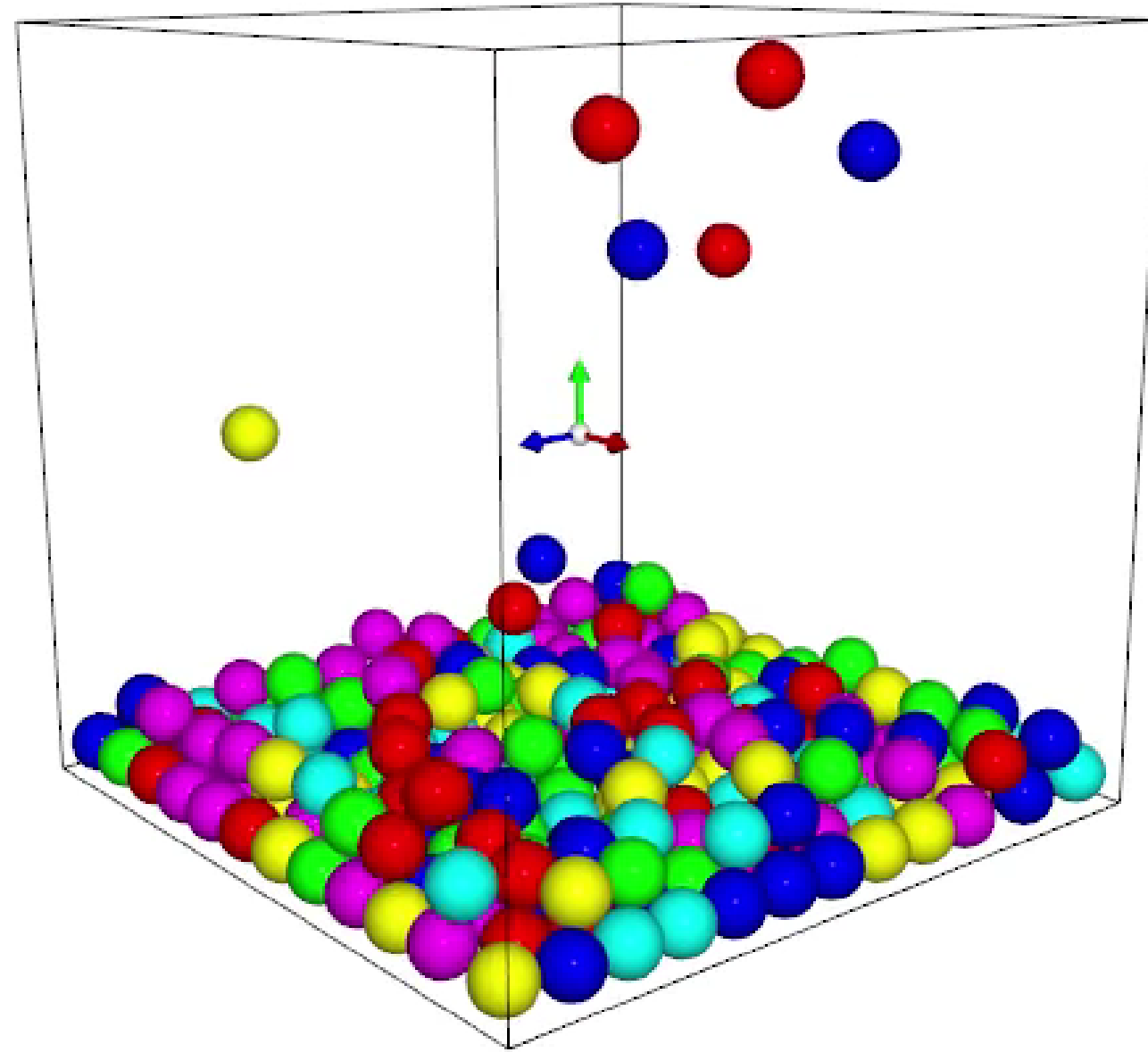


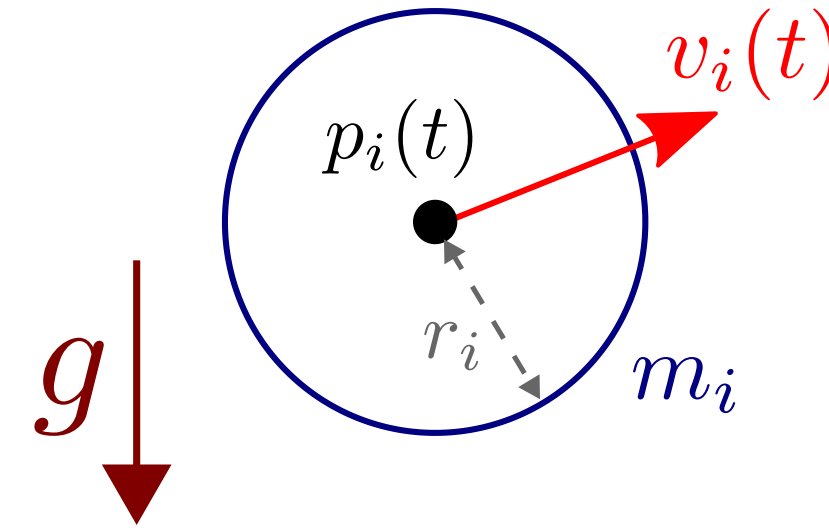
Rigid spheres



System modeling

Particles modeling the center of hard spheres.

- Spheres can collide with surrounding obstacles
- Spheres can collide with each others
- *System*: N particles with position p_i , velocity v_i , mass m_i , modeling a sphere of radius r_i .
 - Initial conditions $p_i(0) = p_i^0, v_i(0) = v_i^0$
- *Forces*: Single gravity forces $F_i = m_i g$. Collisions handled by *impulses*.
- *Temporal evolution*: Fundamental principle of dynamics $v_i(t) = p'_i(t), v'_i(t) = g$
- *Numerical solution*
$$\begin{cases} v^{k+1} = v^k + h g \\ p^{k+1} = p^k + h v^{k+1} \end{cases}$$



Collision with a plane

Plane $\mathcal{P}$: parameterized using a point a and its normal n .

$$\{p \in \mathbb{R}^3 \in \mathcal{P} \Rightarrow (p - a) \cdot n = 0\}$$

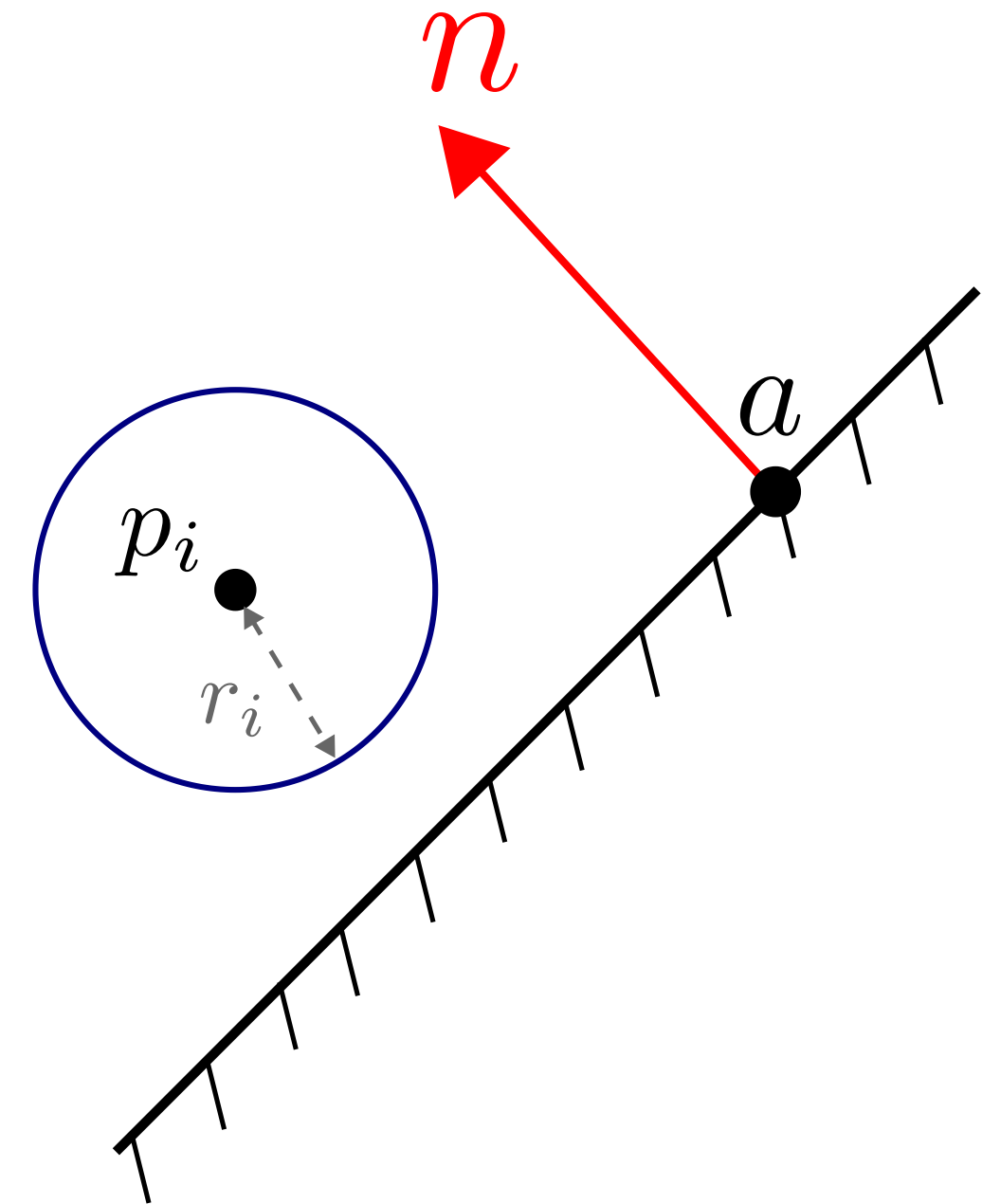
- Sphere above plane : $(p_i - a) \cdot n > r_i$

- Sphere in collision: $(p_i - a) \cdot n \leq r_i$

- Collision detection algorithm

```
for(int i=0; i<N; ++i)
{
    float detection = dot(p[i]-a, n);
    if (detection <= r[i])
    {
        // ... collision response
    }
}
```

What should we do when a collision is detected



Collision response with plane

Suppose exact contact: $(p_i - a) \cdot n = r_i$

Collision response = **Update velocity**

Split $v = v_{//} + v_{\perp}$

$$- v_{\perp} = (v \cdot n) n$$

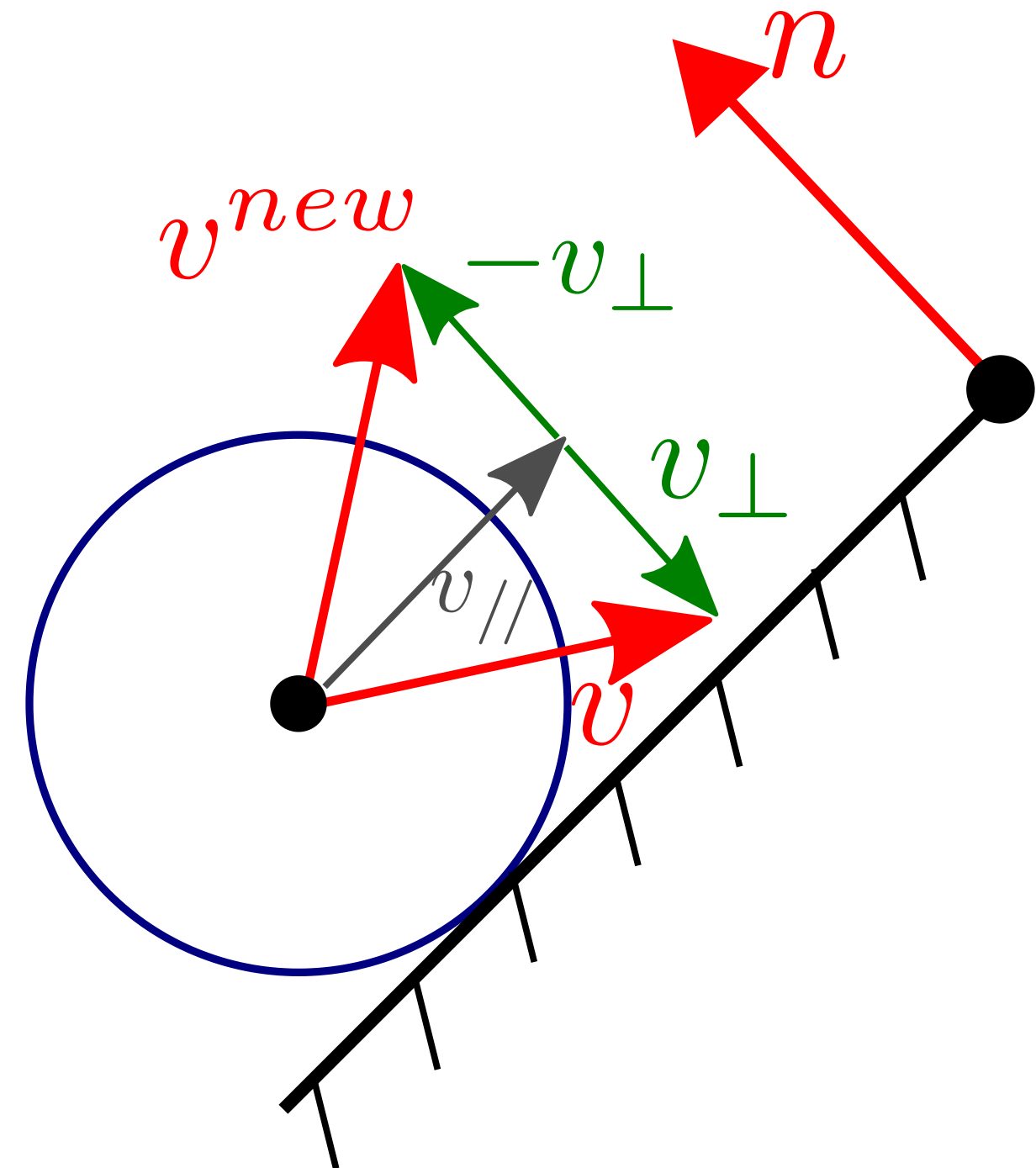
$$- v_{//} = v - (v \cdot n)n$$

New velocity

$$v^{new} = \alpha v_{//} - \beta v_{\perp}$$

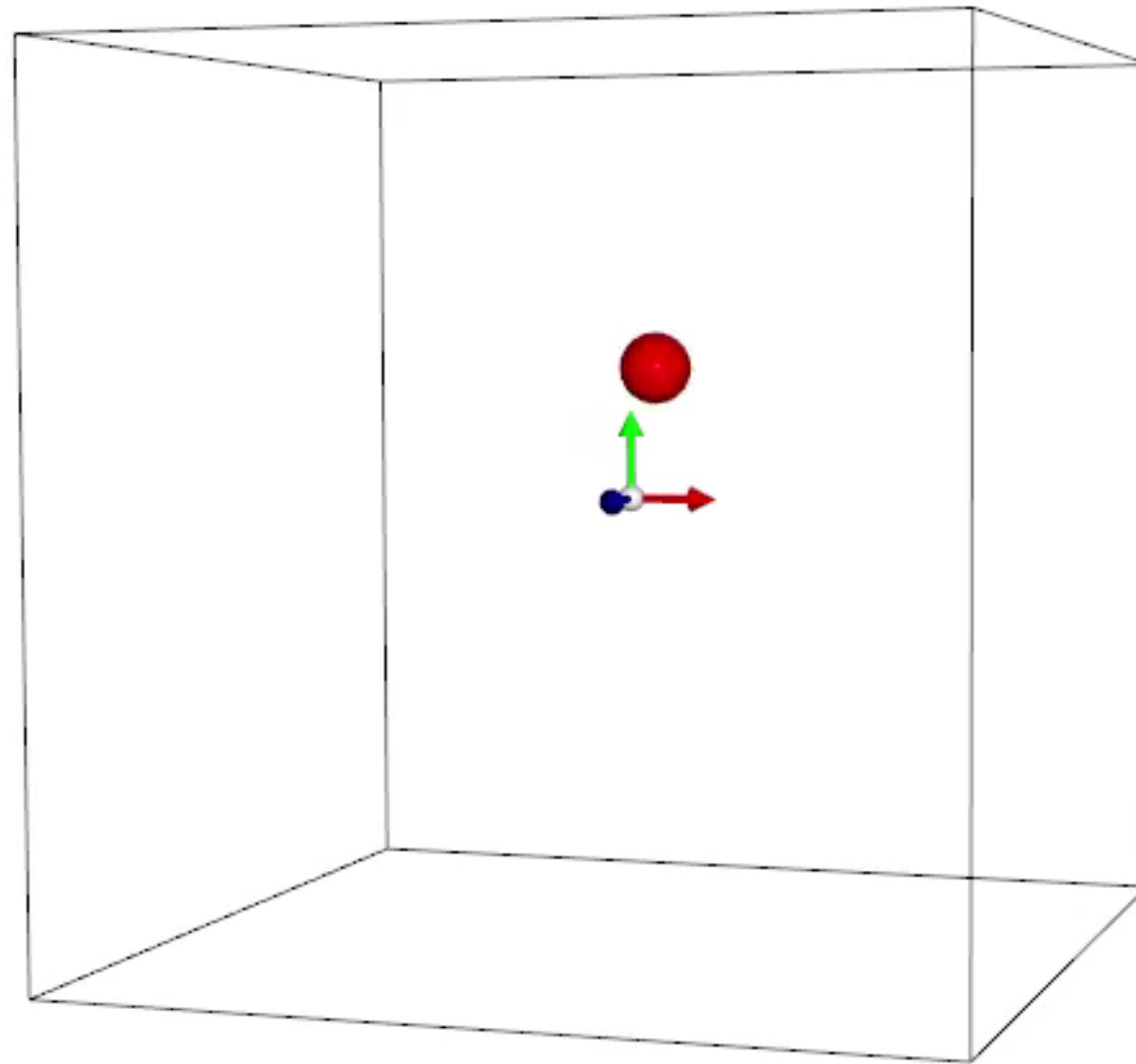
$\alpha \in [0, 1]$ Restitution coefficient in $//$ direction (friction)

$\beta \in [0, 1]$ Restitution coefficient in $\perp$ direction (impact)



Result: Collision response

Applying collision response on speed only

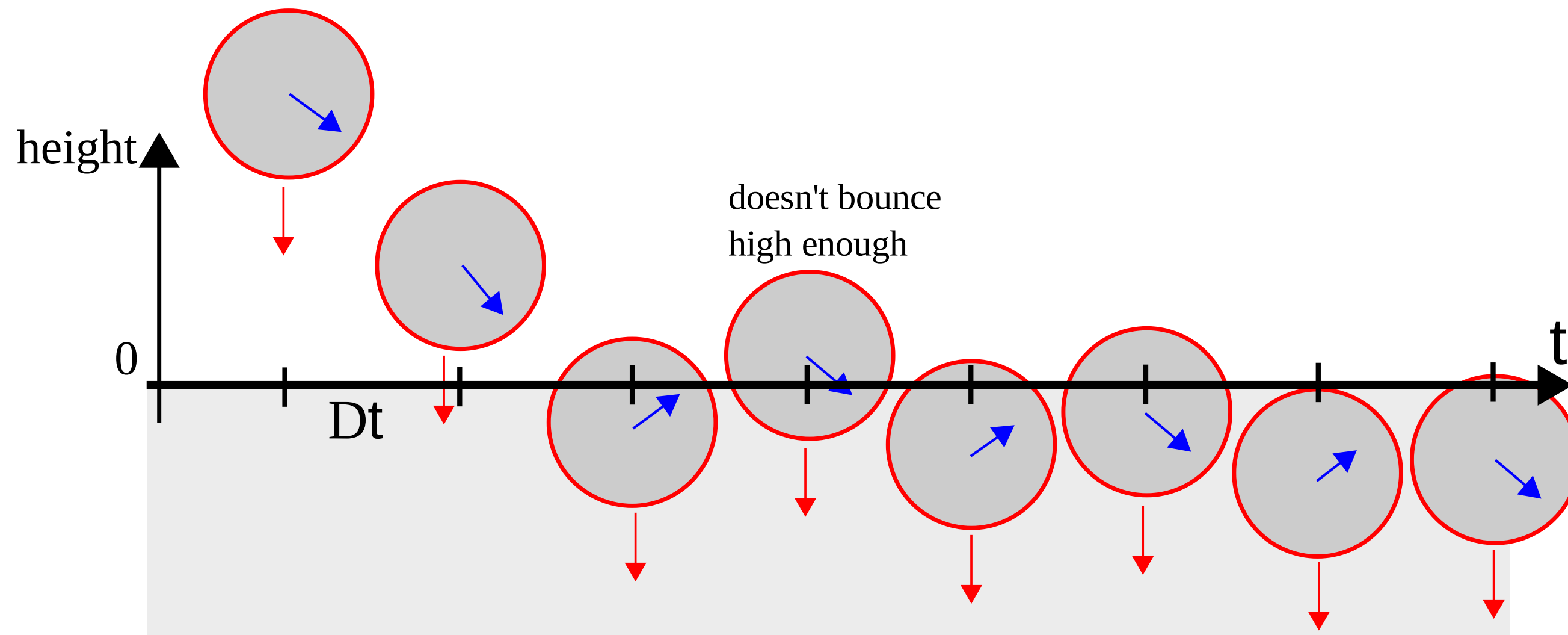


Result: Collision response - issue with discrete time

We assumed contact b/w sphere and plane

But: Exact contact never happens in discrete time

- When collision is detected $\rightarrow$ already inside the wall
- Weight is still acting



Collision response with plane : position

In real case (discrete time) no exact contact, but penetration $(p_i - a) \cdot n_i < r_i$
 $\Rightarrow$ Need to compute collision response at contact point.

Three possibilities

(1) Correct position in projecting on the
constraint

(+) *Simple to implement*

(-) *Physically incorrect position*

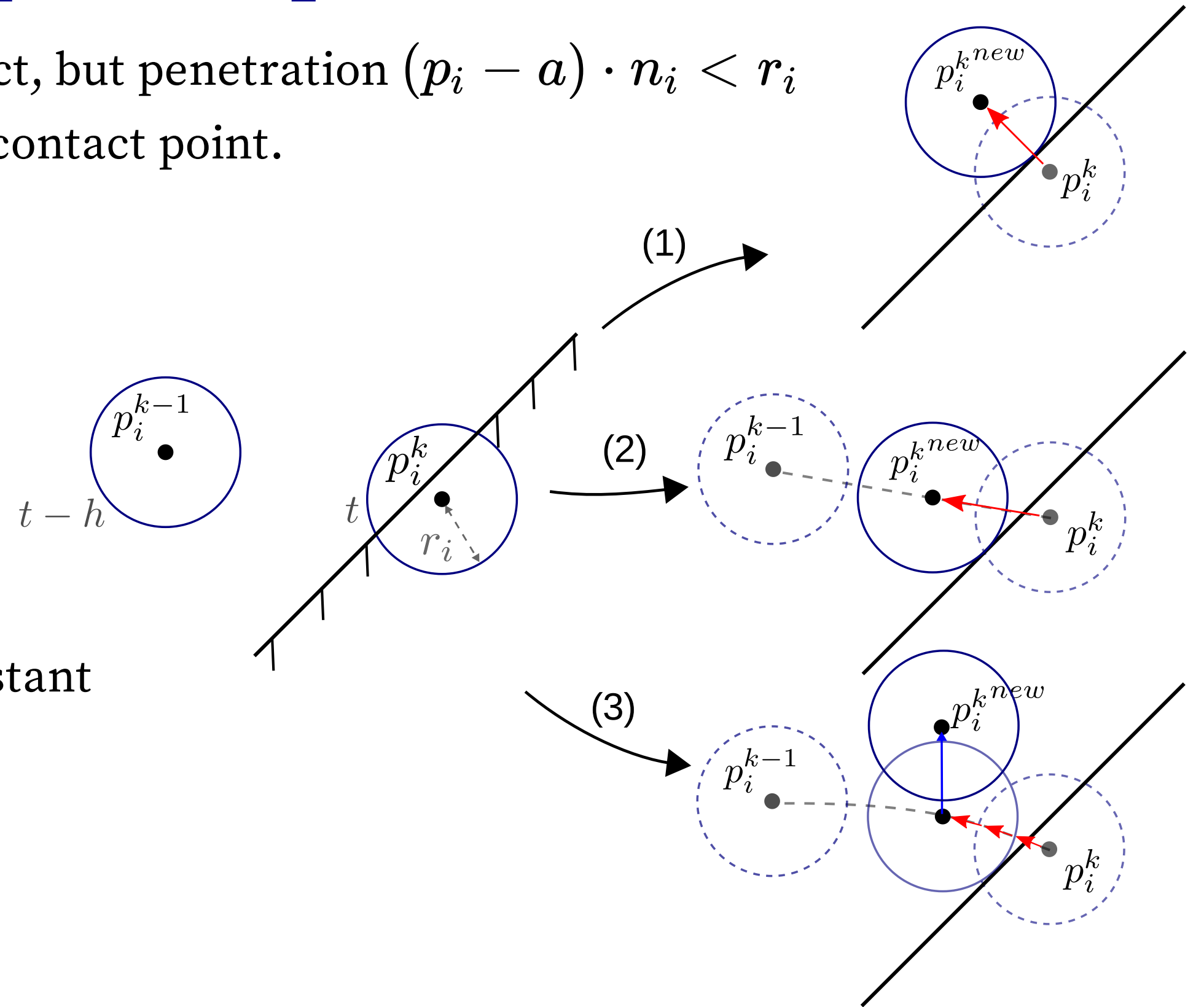
(2) Approximate the correct position

(3) Go backward in time to find exact instant
of collision

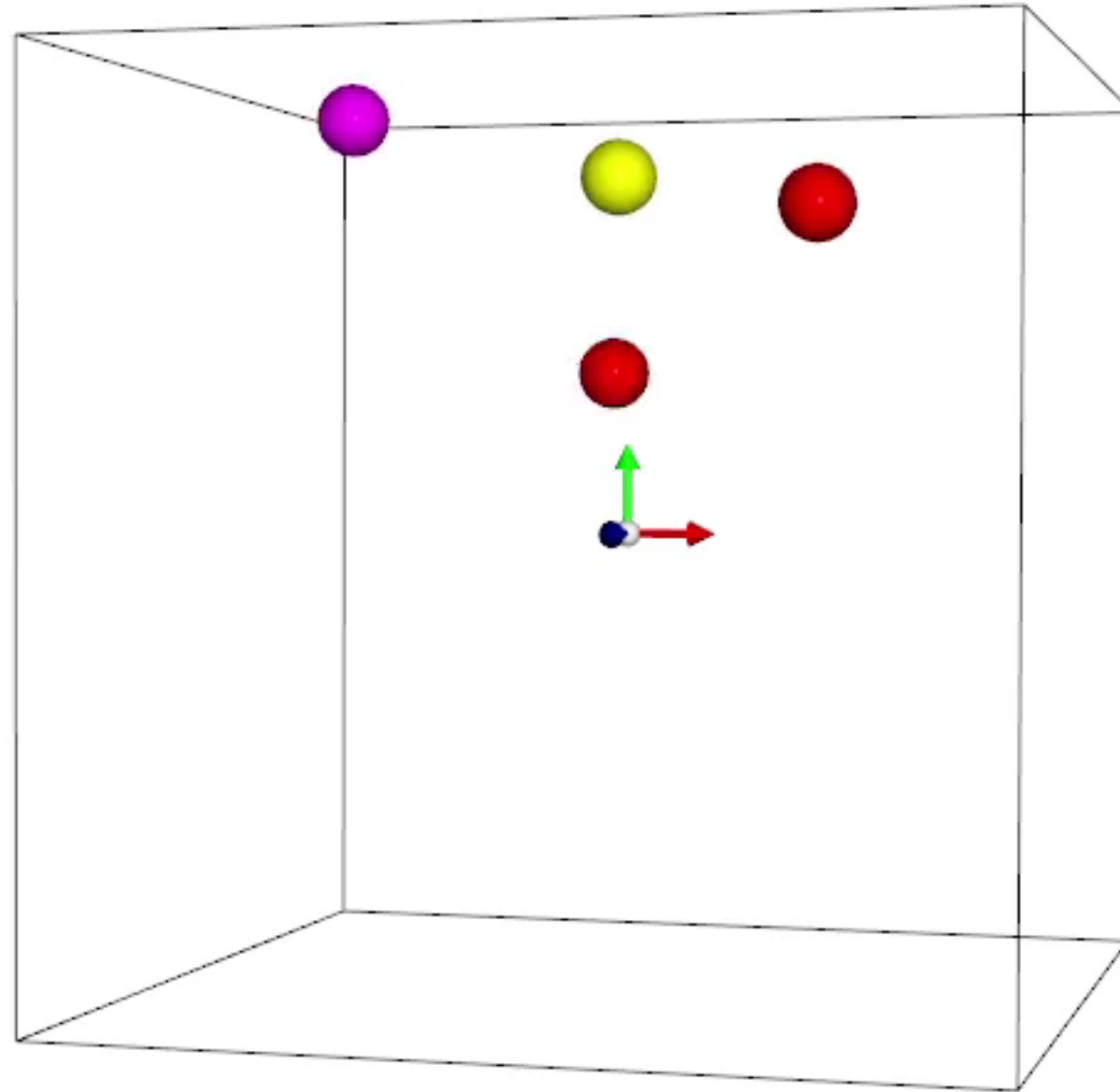
Continuous Collision Detection

(+) *Physically correct*

(-) *Computationally heavy (binary search, etc.)*



Result: Projecting position on plane



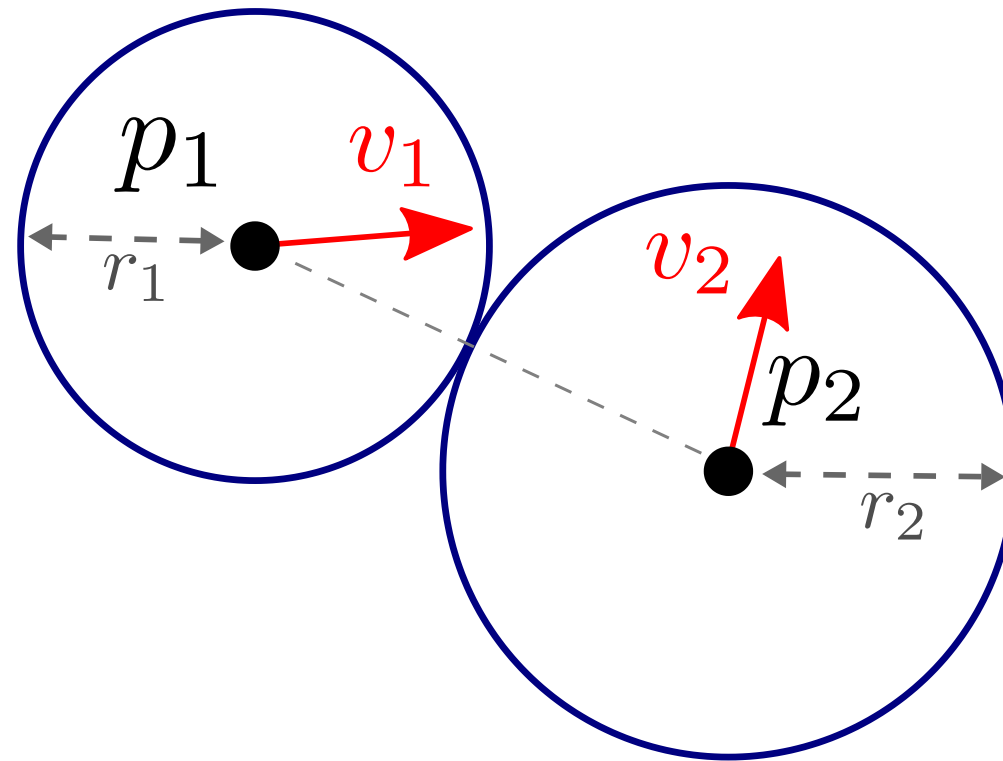
$$p_i^{new} = p_i + d n$$

$$d = r_i - (p_i - a) \cdot n_i : \text{distance of penetration}$$

Collision between spheres

Given 2 spheres $(p_1, v_1, r_1, m_1), (p_2, v_2, r_2, m_2)$.

Collision when $\|p_1 - p_2\| \leq r_1 + r_2$



What happen with their velocities ?

$$v_1 \rightarrow v_1^{new}, v_2 \rightarrow v_2^{new}$$

Notion of impulse

An impulse J is the integrated force over time $J = \int_{t_1}^{t_2} F(t) dt$

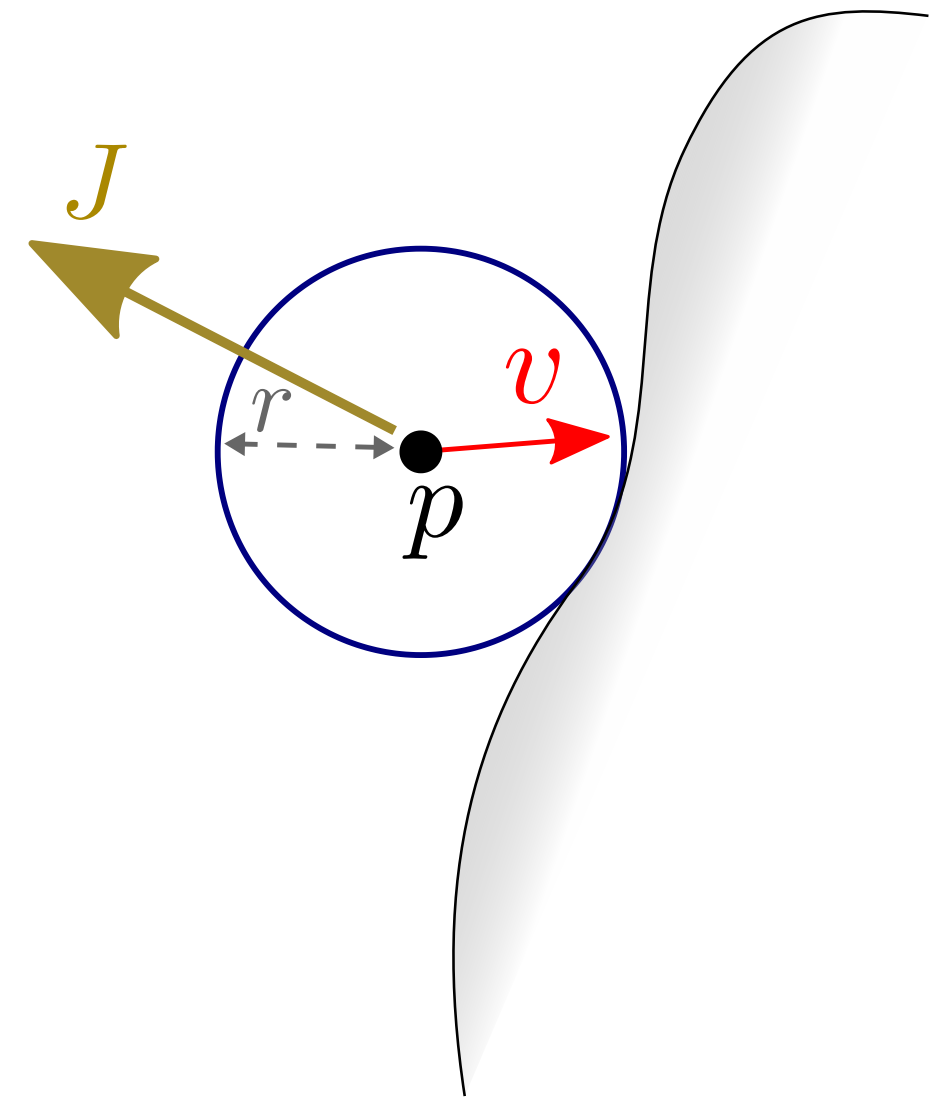
→ results in a sudden change of speed (/momentum) in a discrete case

For a particle with constant mass

$$\int_{t_1}^{t_2} F(t) dt = \int_{t_1}^{t_2} m a(t) dt$$
$$\Rightarrow J = m (v(t_2) - v(t_1))$$

For an impact $v \rightarrow v^{new}$

$$v^{new} = v + J/m$$



Two spheres in collision

Impulse orthogonal to the separating plane between the two surfaces

$$J = j u, \quad u = (p_1 - p_2) / \|p_1 - p_2\|$$

The system with the two spheres is preserving its linear momentum

$\Rightarrow$ Respective impulses j are equals in magnitude, and opposed in direction

$$m_1 v_1 + m_2 v_2 = m_1 v_1^{new} + m_2 v_2^{new} \Rightarrow m_1 (v_1^{new} - v_1) = -m_2 (v_2^{new} - v_2) \Rightarrow J_1 = -J_2$$

Assume collision of "hard spheres" = "Elastic collision"

= No loss of energy, conservation of kinetic energy of the system

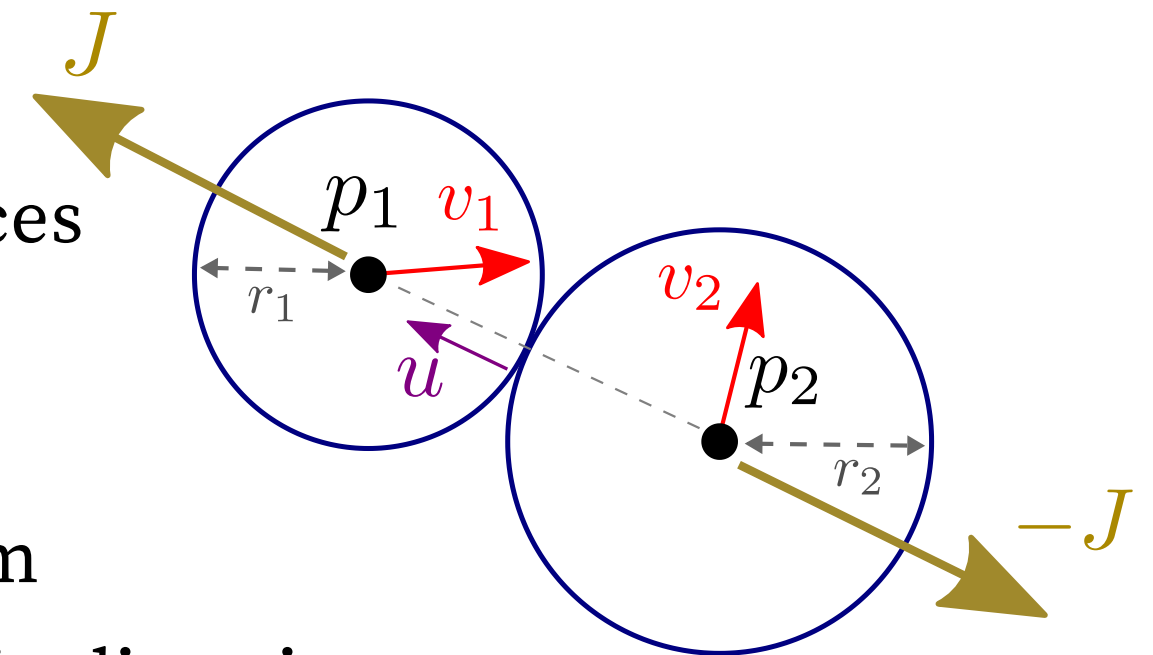
$$\Rightarrow j = 2 \frac{m_1 m_2}{m_1 + m_2} (v_2 - v_1) \cdot u$$

$$1/2 m_1 v_1^2 + 1/2 m_2 v_2^2 = 1/2 m_1 (v_1^{new})^2 + 1/2 m_2 (v_2^{new})^2$$

$$\Rightarrow m_1 v_1^2 + m_2 v_2^2 = m_1 \left(v_1 + \frac{j}{m_1} u \right)^2 + m_2 \left(v_2 - \frac{j}{m_2} u \right)^2$$

$$\Rightarrow 0 = 2 j v_1 \cdot u + \frac{j^2}{m_1} - 2 j v_2 \cdot u + \frac{j^2}{m_2}$$

$$\Rightarrow j = \frac{2}{1/m_1 + 1/m_2} (v_2 - v_1) \cdot u$$



Two spheres in collision

$$v_1^{new} = v_1 + j/m_1 u = v_1 + 2 \frac{m_2}{m_1 + m_2} ((v_2 - v_1) \cdot u) u$$

$$v_2^{new} = v_2 - j/m_2 u = v_2 - 2 \frac{m_1}{m_1 + m_2} ((v_2 - v_1) \cdot u) u$$

Rem. If $m_1 = m_2$: Switch their $\perp$ speeds

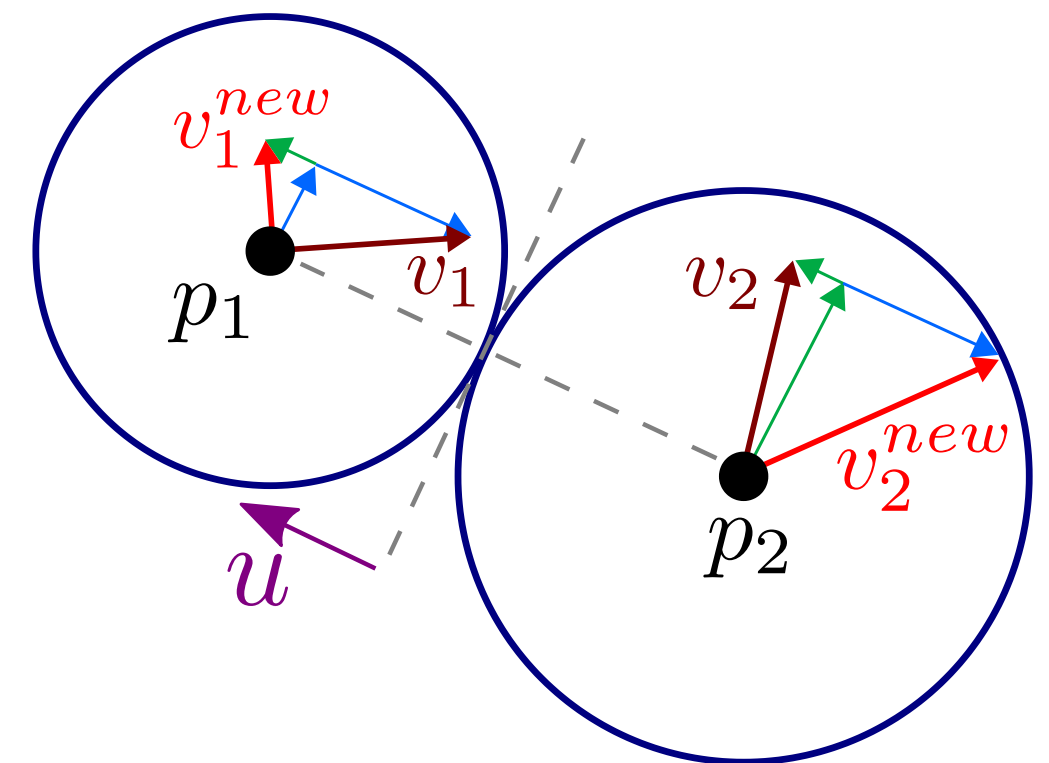
$$v_1^{new} = v_1 + ((v_2 - v_1) \cdot u) u = v_{1//} + v_{2\perp}$$

$$v_2^{new} = v_2 - ((v_2 - v_1) \cdot u) u = v_{2//} + v_{1\perp}$$

Can use restitution coefficient and attenuation $(\alpha, \beta) \in [0, 1]$

$$v_1^{new} = \alpha v_{1//} + \beta v_{2\perp}$$

$$v_2^{new} = \alpha v_{2//} + \beta v_{1\perp}$$



Summary

1. Detect collision $\|p_1 - p_2\| \leq r_1 + r_2$

2-a. If collision (relative speed $> \epsilon$)

Elastic collision (/bouncing) $v_{1/2} = \alpha v_{1/2} \pm \beta J / m_{1/2}$

2-b. If *static* contact (relative speed $\leq \epsilon$)

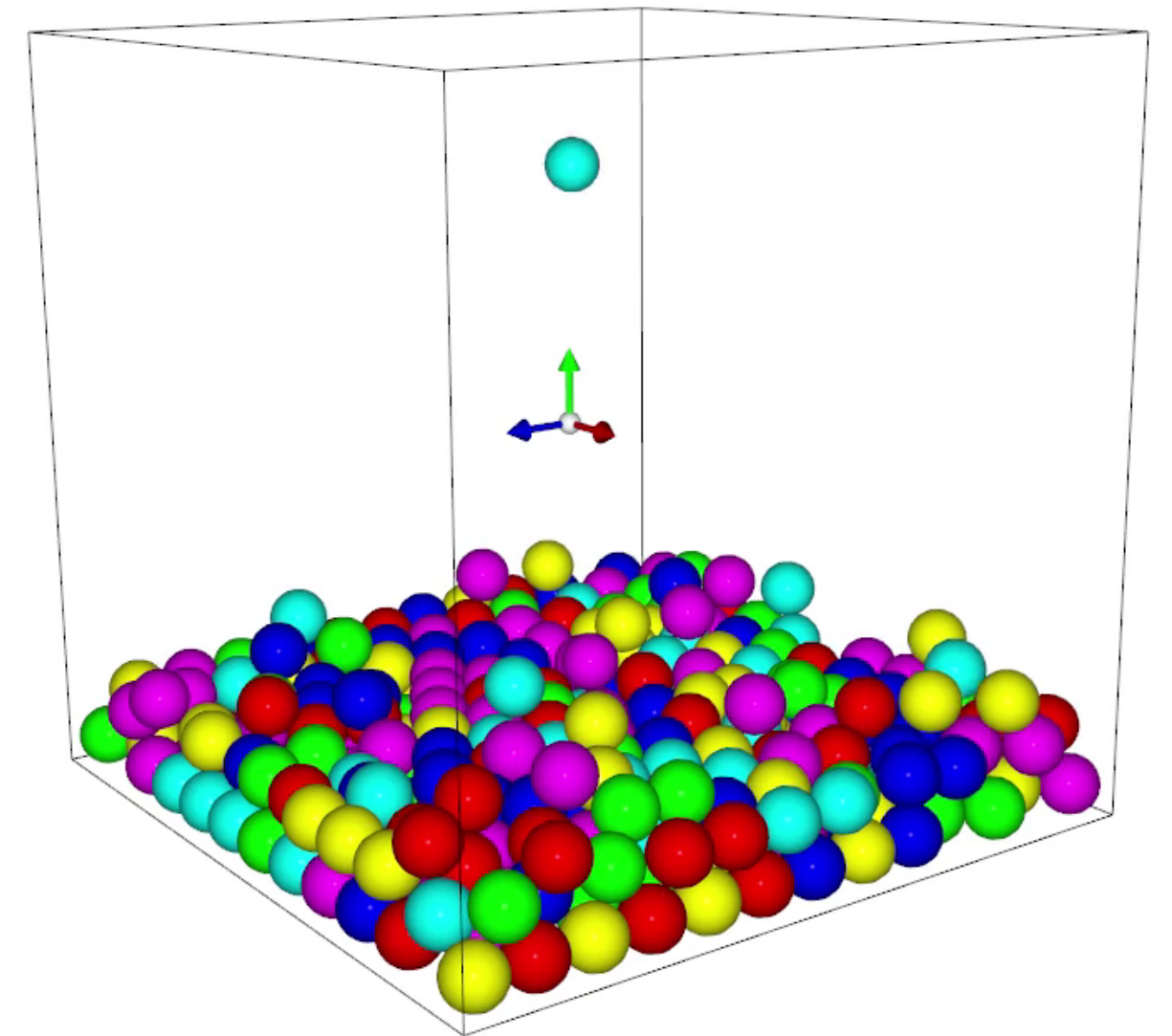
Friction $v_{1/2} = \mu v_{1/2}$, $\mu \in [0, 1]$

Avoids jittering

3. Correct position (project on contact surface)

$$p = p + d/2 u$$

$d = r_1 + r_2 - \|p_1 - p_2\|$: Collision depth

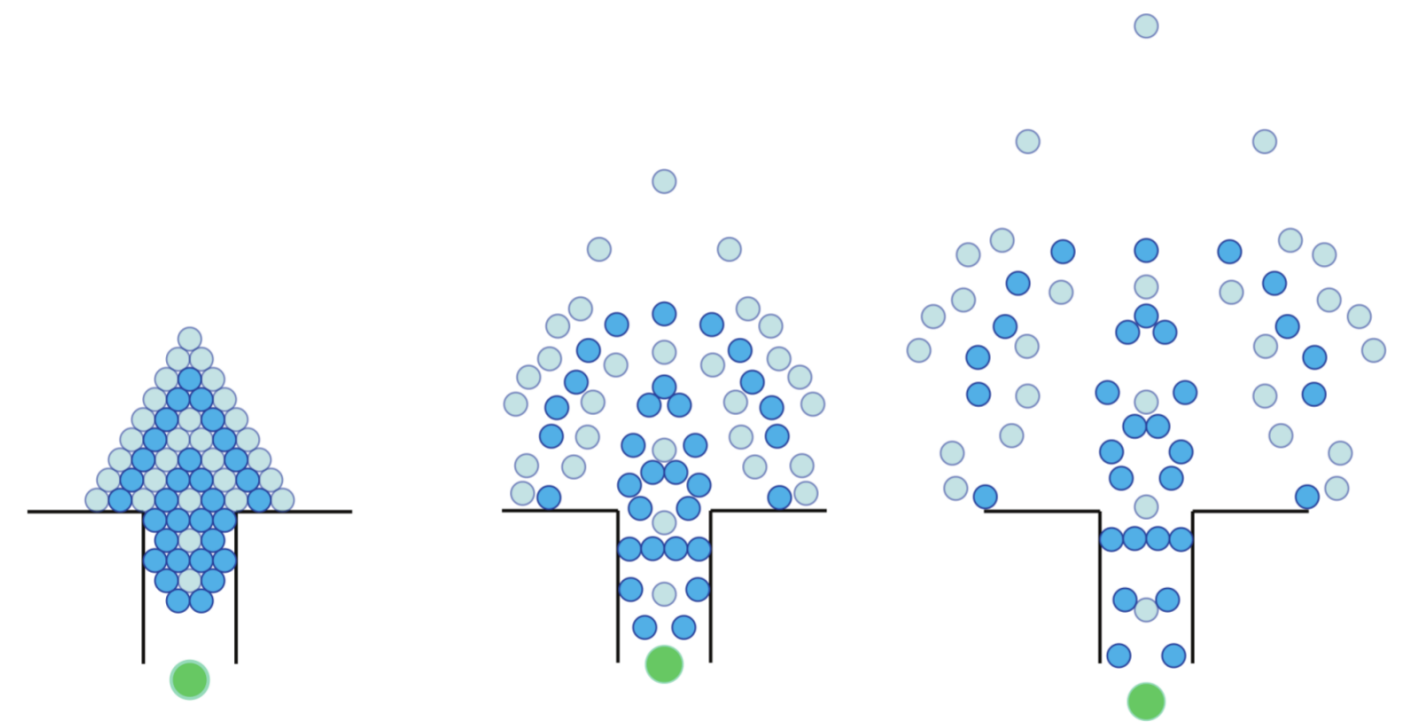


Multiple collisions

Pairwise collisions $\Rightarrow$ no global collision free state

- Correcting one collision may induce new collisions.
- Order of correction does matter

Reducing time-step help, Iterating over multiple pass help



But correct solution in all cases is complex $\rightarrow$ global approach

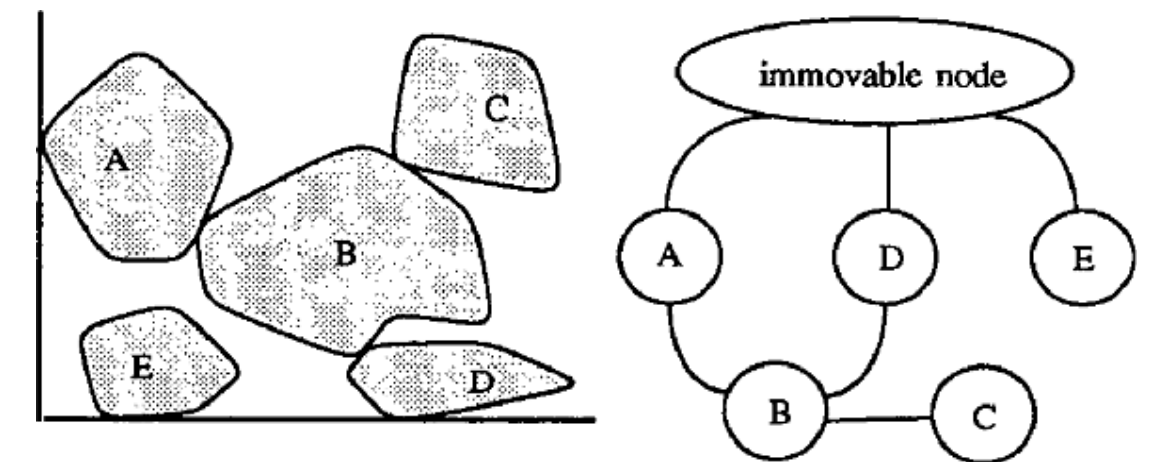
- Precompute contact graph
explicit shock propagation management
- Global constraint-based method

Impulse: $n_i \cdot (v_i - v_j) \geq 0$

Momentum preservation: $m_i v_i - m_j v_j = 0$

Energy preservation/dissipation

$\Rightarrow$ Linear Complementarity Program, Gauss Seidel, etc.



[*Realistic Animation of Rigid Bodies*. J. Hahn. SIGGRAPH 1988.]

[*Collision Detection and Response for Computer Animation*. M. Moore and J. Wilhelms. Computer Graphics 1988.]

[*Reflections on Simultaneous Impact*. B. Smith et al. SIGGRAPH 2012]

[*Guaranteed Resolution of Simultaneous Rigid Body Impact*. E. Vouga. ACM SIGGRAPH 2017]