

Rotations

Rotations

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3D rotations have 3 dof, No unique representation

Matrix

$$R = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

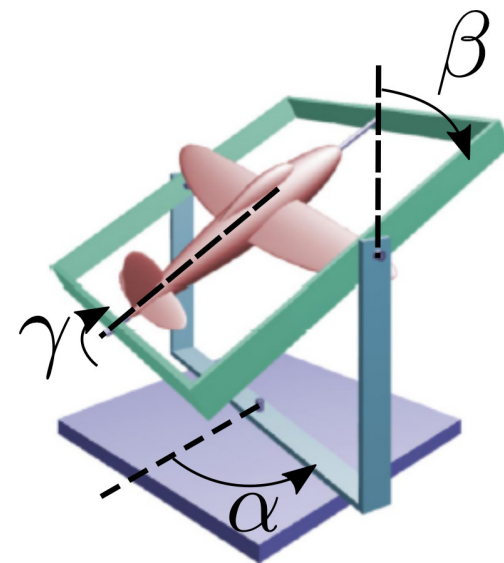
$$R^T R = I$$

$$\det(R) = 1$$

Euler Angles

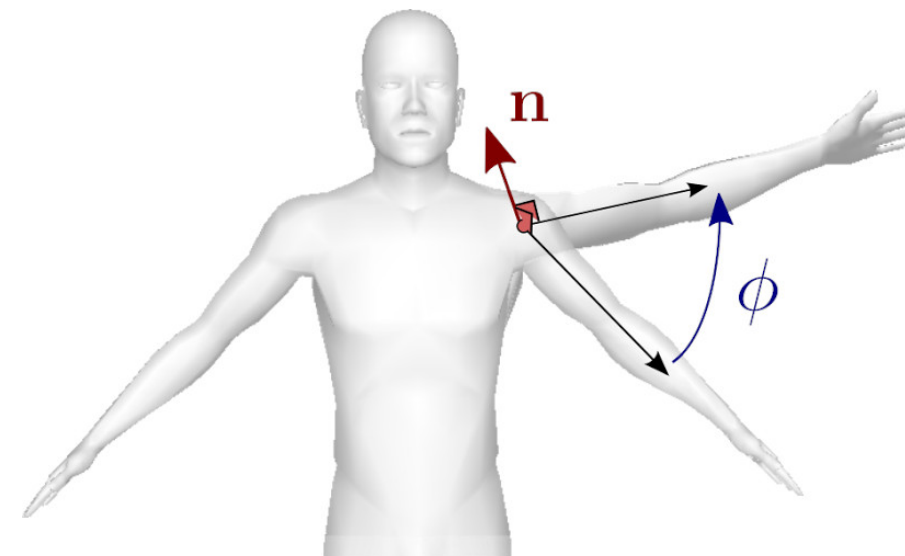
3 angles: (α, β, γ)

Composition of rotation
around basic axes



Axis angle

$(\mathbf{n}, \theta)$



Quaternion

$$q = (x, y, z, w) \\ = \left(\mathbf{n} \sin \left(\frac{\theta}{2} \right), \cos \left(\frac{\theta}{2} \right) \right)$$

Rotation: Focus Euler angles

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Three consecutive rotations along fixed axis along (x, y, z) coordinates.

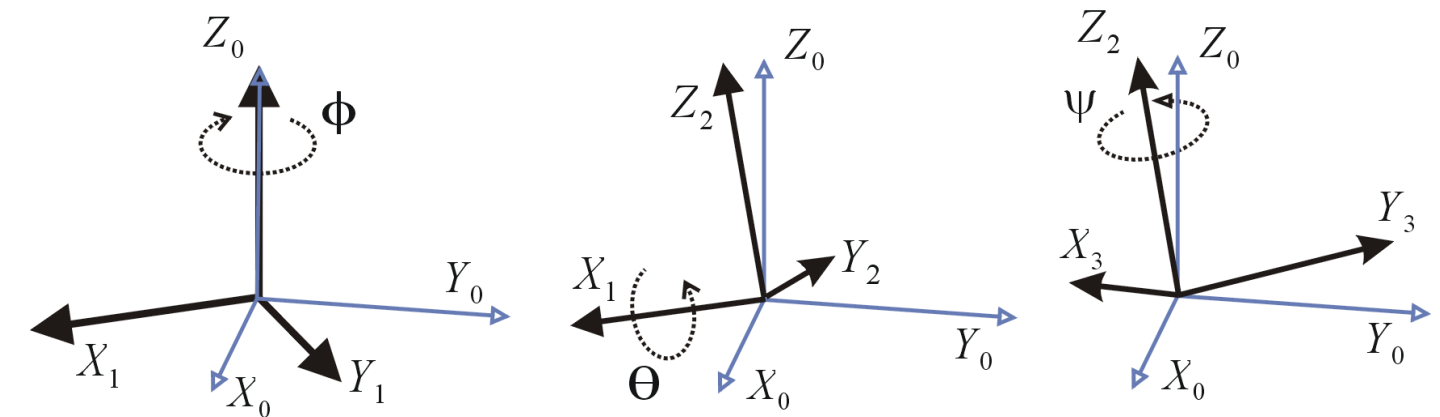
Can use basic rotation matrices composition

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad R_y = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix} \quad R_z = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiple Euler angles conventions

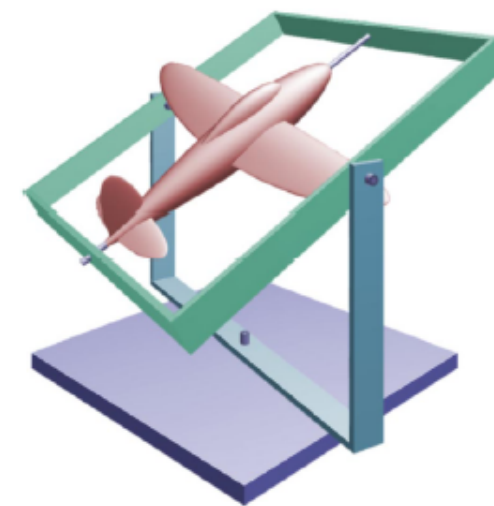
- *Proper Euler* : z - x - z' , x - y - x' , y - z - y' , ...
- *Tait-Bryan* : x - y - z , z - y - x , x - z - y , ...

Take care when exporting/importing/parsing between Softwares



Pro

- Combination of rotation around known axis
- Comprehensive parameters (3 dof)
- Animators can interact with angular curves
- *Widely used in robotics*



Rotation: Focus Euler angles

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Limitations of Euler Angle

- *Gimbal Lock* when composing b/w some rotations

Loose one degree of freedom in specific configuration

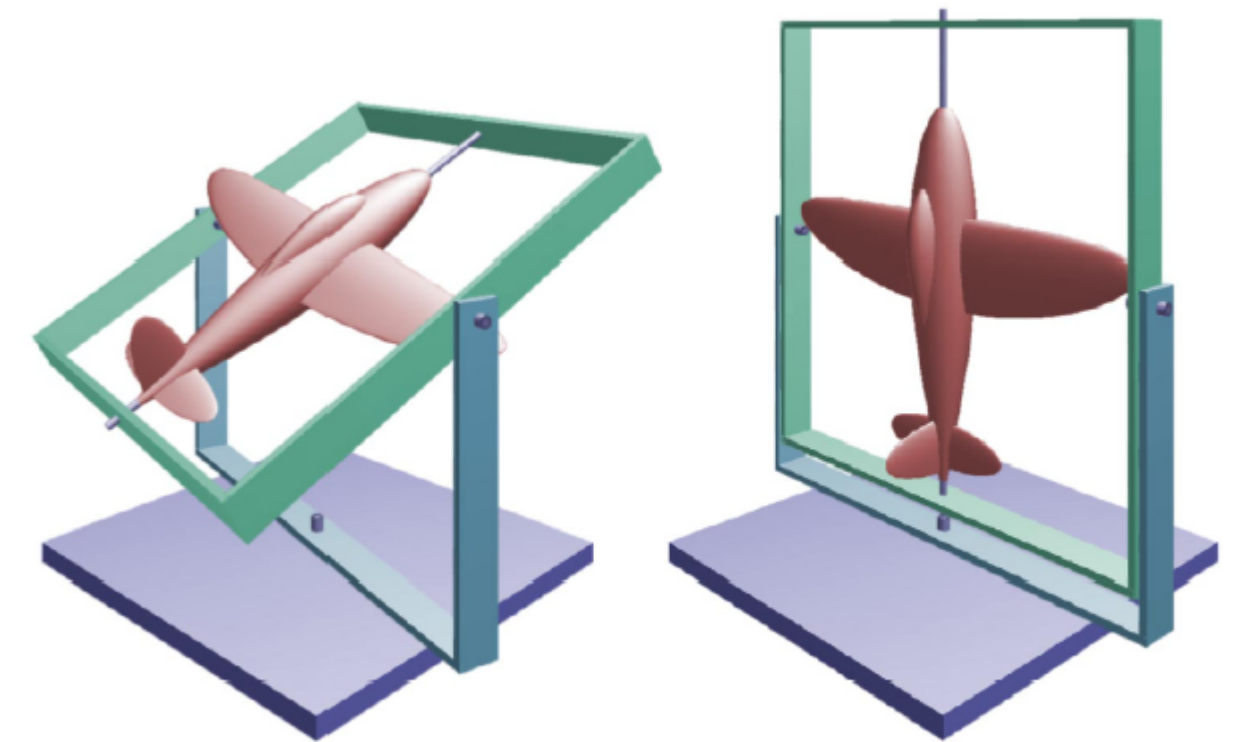
$$\text{ex. } R_x(\alpha) R_y(\pi/2) R_z(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha) \cos(\gamma) + \cos(\alpha) \sin(\gamma) & -\sin(\alpha) \sin(\gamma) + \cos(\alpha) \cos(\gamma) & 0 \\ -\cos(\alpha) \cos(\gamma) + \sin(\alpha) \sin(\gamma) & \cos(\alpha) \sin(\gamma) + \sin(\alpha) \cos(\gamma) & 0 \end{pmatrix}$$

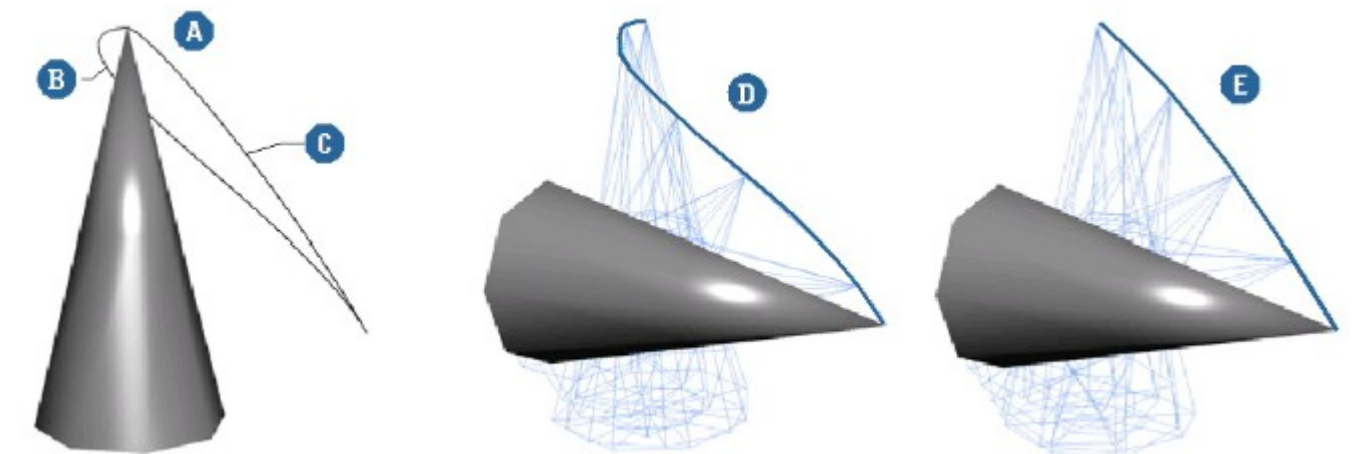
$$= \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ -\cos(\alpha + \gamma) & \sin(\alpha + \gamma) & 0 \end{pmatrix}$$

Expect 2-dof, but get 1-dof

- Interpolation of 3 angles possible but not necessarily with the simplest trajectory.



<http://www.fho-emden.de/~hoffmann/gimbal09082002.pdf>

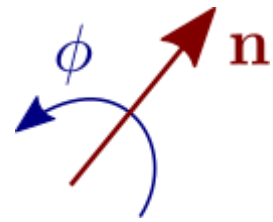


Rotation: Focus Axis Angle

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Any 3D rotation can be represented by

- A unit axis n
- An angle θ

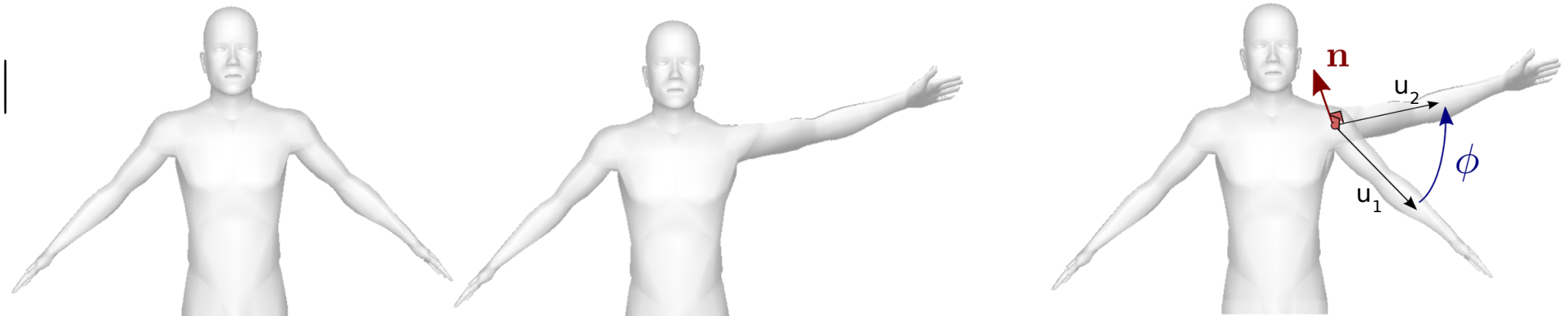


Pro.

- Concise representation: 3 DOF
- Meaningfull parameters
- Describes well the rotation between two vectors

Ex. Rotation between u_1 and u_2

- Axis of rotation $n = (u_1 \times u_2) / \|u_1 \times u_2\|$
- Angle of rotation $\theta = \text{acos}(u_1 \cdot u_2)$
- Twist around u_2 can be arbitrarily chosen



Rotation: Focus Axis Angle - Rodrigues

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Applying a rotation (n, θ) to a vector v

$$v = v_{\parallel} + v_{\perp}$$

$$v' = v'_{\parallel} + v'_{\perp}$$

$$\Rightarrow v' = v_{\parallel} + (\cos(\theta) v_{\perp} + \sin(\theta) n \times v_{\perp})$$

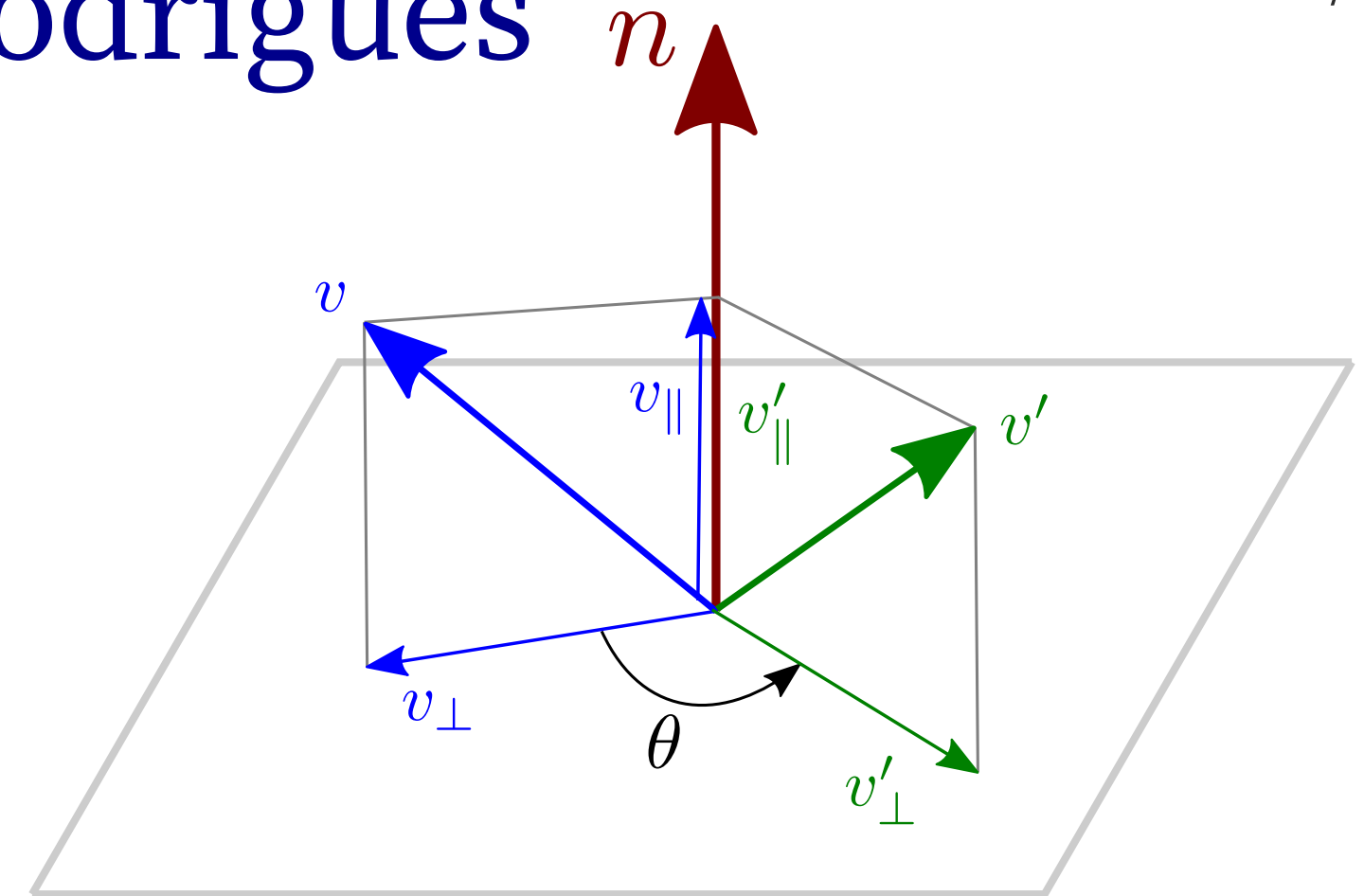
$$v_{\parallel} = (v \cdot n) n$$

$$v_{\perp} = v - (v \cdot n) n$$

$$v' = (v \cdot n) n + \cos(\theta)(v - (v \cdot n) n) + \sin(\theta) n \times (v - (v \cdot n) n)$$

$$\Rightarrow v' = \cos(\theta) v + \sin(\theta) n \times v + (1 - \cos(\theta)) (v \cdot n) n$$

Rodrigues' rotation Formula



Rotation: Focus Axis Angle as Matrix

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$$v' = \cos(\theta) v + \sin(\theta) n \times v + (1 - \cos(\theta)) (v \cdot n) n = R(n, \theta) v$$

$$v' = \cos(\theta) v + \sin(\theta) \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \times v + (1 - \cos(\theta)) \begin{pmatrix} n_x & n_y & n_z \end{pmatrix} v \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

$$v' = \cos(\theta) v + \sin(\theta) \underbrace{\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}}_K v + (1 - \cos(\theta)) \underbrace{\begin{pmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{pmatrix}}_{K^2} v$$

$$v' = \begin{pmatrix} \cos(\theta) + n_x^2(1 - \cos(\theta)) & n_x n_y(1 - \cos(\theta)) - n_z \sin(\theta) & n_x n_z(1 - \cos(\theta)) + n_y \sin(\theta) \\ n_x n_y(1 - \cos(\theta)) + n_z \sin(\theta) & \cos(\theta) + n_y^2(1 - \cos(\theta)) & n_y n_z(1 - \cos(\theta)) - n_x \sin(\theta) \\ n_x n_z(1 - \cos(\theta)) - n_y \sin(\theta) & n_y n_z(1 - \cos(\theta)) + n_x \sin(\theta) & \cos(\theta) + n_z^2(1 - \cos(\theta)) \end{pmatrix} v$$

Rotation: Focus Axis Angle - Summary

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Given a rotation (n, θ) - Corresponding rotation matrix

$$R(n, \theta) = I + \sin(\theta) K + (1 - \cos(\theta)) K^2$$

$$R(n, \theta) = \begin{pmatrix} \cos(\theta) + n_x^2(1 - \cos(\theta)) & n_x n_y(1 - \cos(\theta)) - n_z \sin(\theta) & n_x n_z(1 - \cos(\theta)) + n_y \sin(\theta) \\ n_x n_y(1 - \cos(\theta)) + n_z \sin(\theta) & \cos(\theta) + n_y^2(1 - \cos(\theta)) & n_y n_z(1 - \cos(\theta)) - n_x \sin(\theta) \\ n_x n_z(1 - \cos(\theta)) - n_y \sin(\theta) & n_y n_z(1 - \cos(\theta)) + n_x \sin(\theta) & \cos(\theta) + n_z^2(1 - \cos(\theta)) \end{pmatrix}$$

Pro

- Concise and general representation
- Expressive parameters in 3D space (axe, angles)
- Efficient rotation and correspondance with matrix

Cons

- No simple composition expression between two rotations.
- No direct interpolation

(well handled by quaternion representation)

Rotation: Focus on Quaternions

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Quaternions: generalization of complex numbers.

$$q = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} + w \quad w \text{ real part, } (x, y, z) \text{ imaginary (or pure quaternion) part.}$$

We write in short $q = (x, y, z, w)$

(don't confound with 4D vectors in homogeneous coordinates)

Properties of imaginary basis vectors

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

$$\mathbf{ijk} = -1$$

- Provides the algebraic properties

Basics operations on quaternions

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- Conjugated quaternion $q^* = (-x, -y, -z, w)$

- Quaternion norm $\|q\| = \sqrt{q q^*} = \sqrt{x^2 + y^2 + z^2 + w^2}$. Unit quaternion satisfies $\|q\| = 1$.

- Quaternion product

$$q_1 q_2 = (x_1 \mathbf{i} + y_1 \mathbf{j} + z_1 \mathbf{k} + w_1) (x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k} + w_2) = \dots$$

$$q_1 q_2 = \begin{pmatrix} x_1 w_2 + w_1 x_2 + y_1 z_2 - z_1 y_2 \\ y_1 w_2 + w_1 y_2 + z_1 x_2 - x_1 z_2 \\ z_1 w_2 + w_1 z_2 + x_1 y_2 - y_1 x_2 \\ w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 \end{pmatrix}$$

Note: Quaternion product is

- associative: $(q_1 q_2) q_3 = q_1 (q_2 q_3) = q_1 q_2 q_3$
- **non-commutative**: $q_1 q_2 \neq q_2 q_1$

Sometimes interesting to separate *real part* w from *pure quaternion* part $s = (x, y, z)$.

- Shorthand vector form $q = (s, w)$

- Quaternion product in vector form $q_1 q_2 = (s_1 w_2 + s_2 w_1 + s_1 \times s_2, w_1 w_2 - s_1 \cdot s_2)$

Relation between quaternion and rotation

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Consider

- a quaternion $q = (s, w)$ of unit norm $\|q\| = 1$.
- a vector $v = (v_x, v_y, v_z)$ assimilated to the pure quaternion $q_v = (v, 0) = (v_x, v_y, v_z, 0)$.

Then

- $q_{v'} = \mathcal{R}_q(v) = q q_v q^*$ is a pure quaternion $q_{v'} = (v'_x, v'_y, v'_z, 0)$
- And $v' = (v'_x, v'_y, v'_z)$ is the rotation of vector v around the axis $n = s/\|s\|$, with angle $2 \arccos(w)$.

Demonstration

$$\mathcal{R}_q(v) = (s, w) (v, 0) (-s, w) = \dots = ((w^2 - s^2) v + 2(s \cdot v) s + 2w (s \times v), 0)$$

As $\|q\| = 1$, we can write $q = (s, w) = (n \sin(\phi), \cos(\phi))$, where $\|n\| = 1$

$$\text{Then } \mathcal{R}_q(v) = \underbrace{(\cos^2(\phi) - \sin^2(\phi))}_{\cos(2\phi)} v + \underbrace{2 \sin^2(\phi)}_{1 - \cos(2\phi)} (n \cdot v) n + \underbrace{2 \cos(\phi) \sin(\phi)}_{\sin(2\phi)} n \times v, 0)$$

$\Rightarrow$ Rodrigues formula for axis n and angle 2ϕ .

The unit quaternion $q = (n \sin(\theta/2), \cos(\theta/2))$ represents the rotation of angle θ around the axis n .

Composition of rotations

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Consider two rotation (R_1, R_2) associated to their unit quaternions (q_1, q_2) .

The product $q_1 q_2$ represents the composition $R_1 \circ R_2$.

Demonstration

We show that $\mathcal{R}_{q_1 q_2}(v) = \mathcal{R}_{q_1} \circ \mathcal{R}_{q_2}(v)$.

$$\mathcal{R}_{q_1 q_2}(v) = (q_1 q_2) v (q_1 q_2)^*$$

$$\mathcal{R}_{q_1 q_2}(v) = (q_1 q_2) v (q_2^* q_1^*), \text{ as } (q_1 q_2)^* = q_2^* q_1^*$$

$$\mathcal{R}_{q_1 q_2}(v) = q_1 (q_2 v q_2^*) q_1^*$$

$$\mathcal{R}_{q_1 q_2}(v) = q_1 \mathcal{R}_{q_2}(v) q_1^* = \mathcal{R}_{q_1} \circ \mathcal{R}_{q_2}(v)$$

Correspondance quaternion to rotation matrix

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The unit quaternion $q = (x, y, z, w)$ represents the rotation given by the matrix

$$R = \begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{pmatrix}$$

Demonstration

$$v' = \mathcal{R}_q(v) = q q_v q^* = ((w^2 - s^2) v + 2(s \cdot v) s + 2w (s \times v), 0) \quad \text{with } s = (x, y, z)$$

$$v' = (w^2 - x^2 - y^2 - z^2)v + 2 \begin{pmatrix} x & y & z \end{pmatrix} v \begin{pmatrix} x & y & z \end{pmatrix}^T + 2w \begin{pmatrix} x & y & z \end{pmatrix} \times v$$

$$v' = \left((w^2 - x^2 - y^2 - z^2) \mathbf{I} + 2 \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} + 2w \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \right) v$$

$$v' = R v, \text{ with } x^2 + y^2 + z^2 + w^2 = 1$$

Summary - Correspondance quaternion / matrix-vector

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Representation and Operations

	3D space	Quaternion space
<i>Vector</i>	$v = (v_x, v_y, v_z)$	$q_v = (v, 0) = (v_x, v_y, v_z, 0)$
<i>Rotation</i>	R (3×3 matrix)	$q = (x, y, z, w), \ q\ = 1$
<i>Apply rotation to vector</i>	Rv	$q q_v q^\star$
<i>Rotation composition</i>	$R_1 R_2$	$q_1 q_2$

Rotation to Quaternion

Rotation of axis n and angle $\theta \Rightarrow q = (n \sin(\theta/2), \cos(\theta/2))$

Quaternion to Rotation

$$\text{Unit quaternion } q = (x, y, z, w) \Rightarrow R = \begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{pmatrix}$$

Interpolation

Rotations

Interpolating rotation

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Problem: Given 2 rotations r_1, r_2

- Find rotation $r(t)$ st $r(0) = r_1, r(1) = r_2$, and varies smoothly along t

Matrix representation

Componentwise interpolation not adapted

ex. $t R_1 + (1 - t) R_2$ is not a rotation: introduce scaling/shearing

The correct formulation to interpolate on the manifold would be $R_1 \exp(t \log(R_1^T R_2))$

But matrix exponential is complex to compute

Euler angle

Can interpolate the 3 angles separately

(+) Leads to a rotation

(-) Doesn't necessarily follow simplest trajectory

Axis-Angle

No trivial interpolation scheme with different axes

Quaternion

Well adapted (see later)

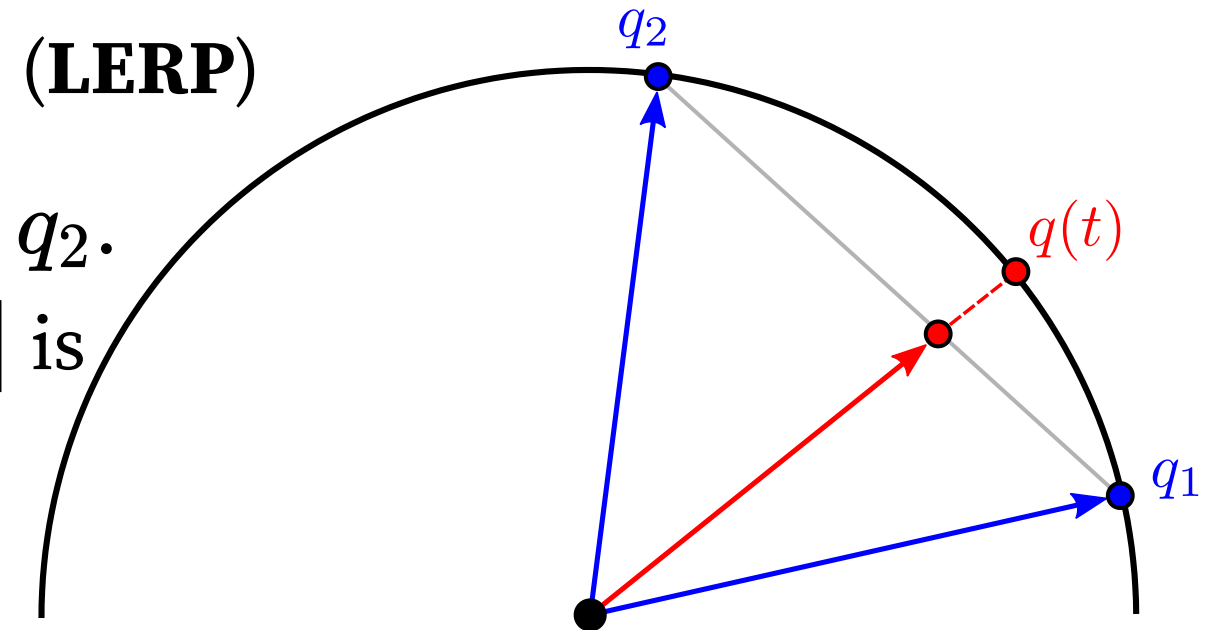
Interpolation of Quaternion - LERP

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Linear interpolation (with normalization) in quaternion space (**LERP**)

- Consider two rotations with corresponding quaternion q_1, q_2 .
- The *linearly interpolated* quaternion at parameter $t \in [0, 1]$ is

$$q(t) = \frac{(1 - t) q_1 + t q_2}{\|(1 - t) q_1 + t q_2\|}$$



- (+) Follows a shortest path (great circle on 4D sphere)
- (+) Can be generalized to more general parametric curves (Splines, etc)
- (-) Angular speed of orientation is not-constant
ex. extreme case of two opposite quaternions

Interpolation of Quaternion - SLERP

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Spherical Linear interpolation (**SLERP**)

- Consider two unit vectors (arbitrary dimensions) v_1, v_2 .
- The *spherical linear interpolation* at parameter $t \in [0, 1]$ is

$$v(t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)} v_1 + \frac{\sin(t\Omega)}{\sin(\Omega)} v_2 \quad \text{with } \cos(\Omega) = v_1 \cdot v_2$$

Demonstration

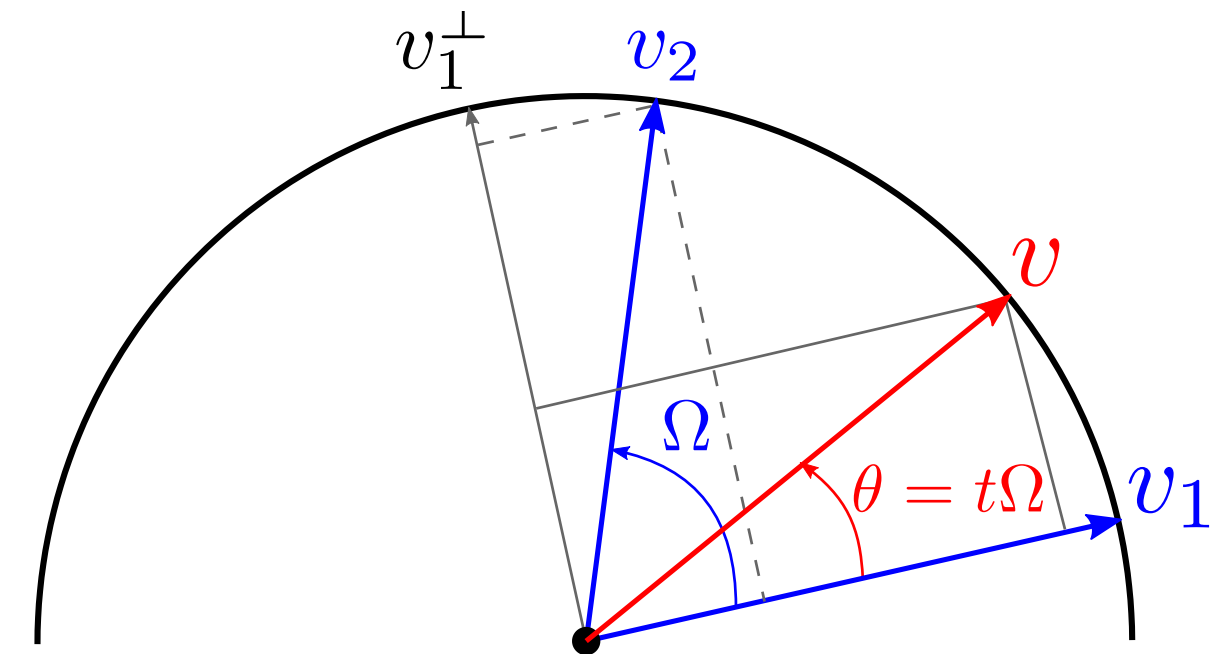
Consider

- Two unit vectors (arbitrary dimensions) v_1, v_2 .
- Ω : angle b/w v_1 and v_2
- The interpolated vector v at angle $\theta = \Omega t, t \in [0, 1]$.

$$v = v_1 \cos(\theta) + v_1^\perp \sin(\theta), \text{ and } v_1^\perp = \frac{v_2 - \cos(\Omega) v_1}{\sin(\Omega)}$$

$$\Rightarrow v = \left(\frac{\sin(\Omega) \cos(\theta) - \cos(\Omega) \sin(\theta)}{\sin(\Omega)} \right) v_1 + \frac{\sin(\theta)}{\sin(\Omega)} v_2$$

$$\Rightarrow v = \frac{\sin(\Omega - \theta)}{\sin(\Omega)} v_1 + \frac{\sin(\theta)}{\sin(\Omega)} v_2$$



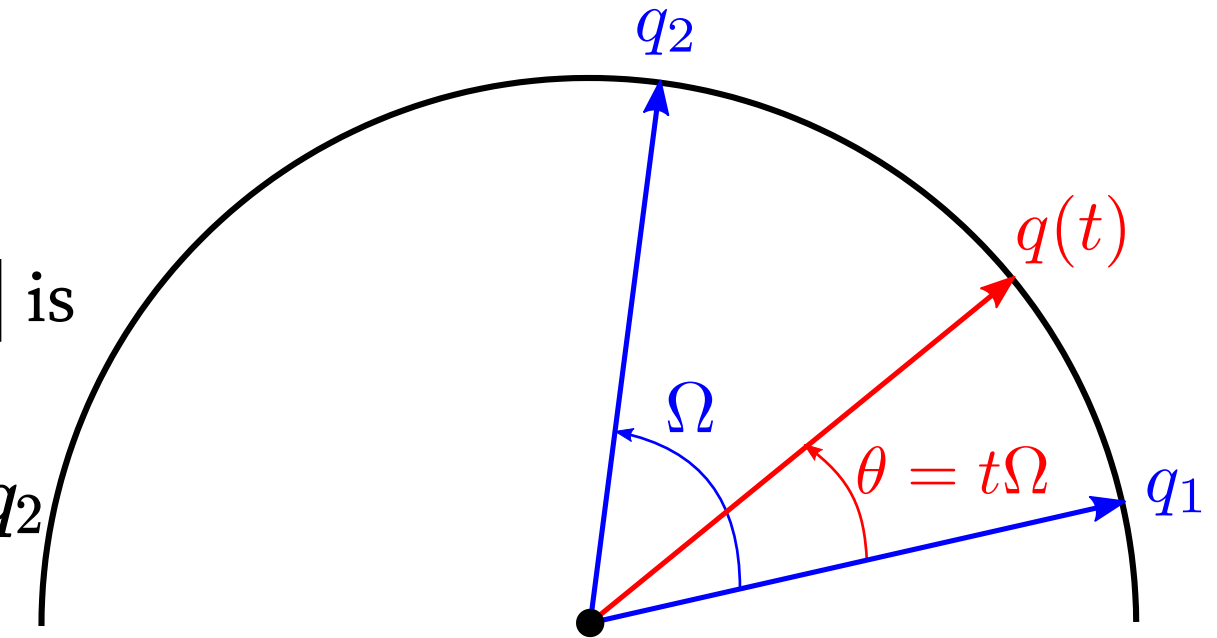
Interpolation of Quaternion - SLERP

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Spherical Linear interpolation (**SLERP**)

- Consider two rotations with corresponding quaternion q_1, q_2 .
- The *spherical linear interpolated* quaternion at parameter $t \in [0, 1]$ is

$$q(t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)} q_1 + \frac{\sin(t\Omega)}{\sin(\Omega)} q_2 \quad \text{with } \cos(\Omega) = q_1 \cdot q_2$$



(+) Follows a shortest path (great circle on 4D sphere)

(+) Constant angular speed

(-) Cannot be generalized to more general curves.

Cannot interpolate between more than two quaternions

Care with quaternion negation

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$+q$ and $-q$ correspond to the same rotation ($n \rightarrow -n, \theta \rightarrow 2\pi - \theta$)

Warning $-q$ **does not** corresponds to the rotation matrix $-R$.

But to a **different path** when interpolated in the 4D quaternion space.

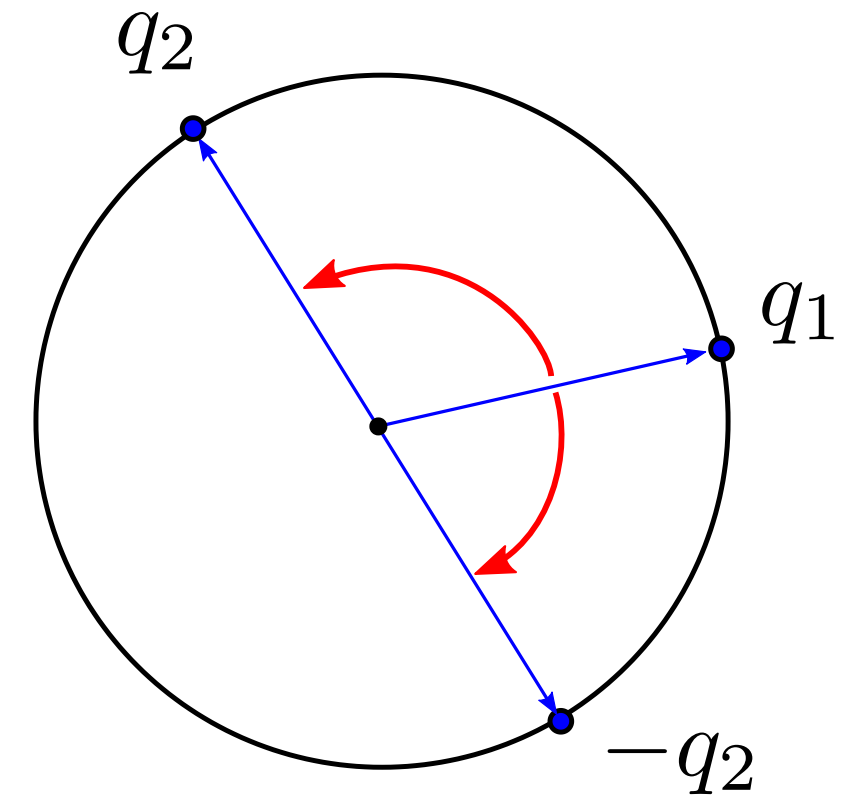
$\Rightarrow$ path $q_1 \rightarrow -q_2$ is shorter than $q_1 \rightarrow q_2$ when $q_1 \cdot q_2 < 0$.

In practice we check for the shorter path before applying SLERP.

Algorithm

```
if( dot(q1,q2)<0 )  
    q2 = -q2
```

```
q(t) = SLERP(q1,q2,t)
```



Interpolation

Affine Transforms

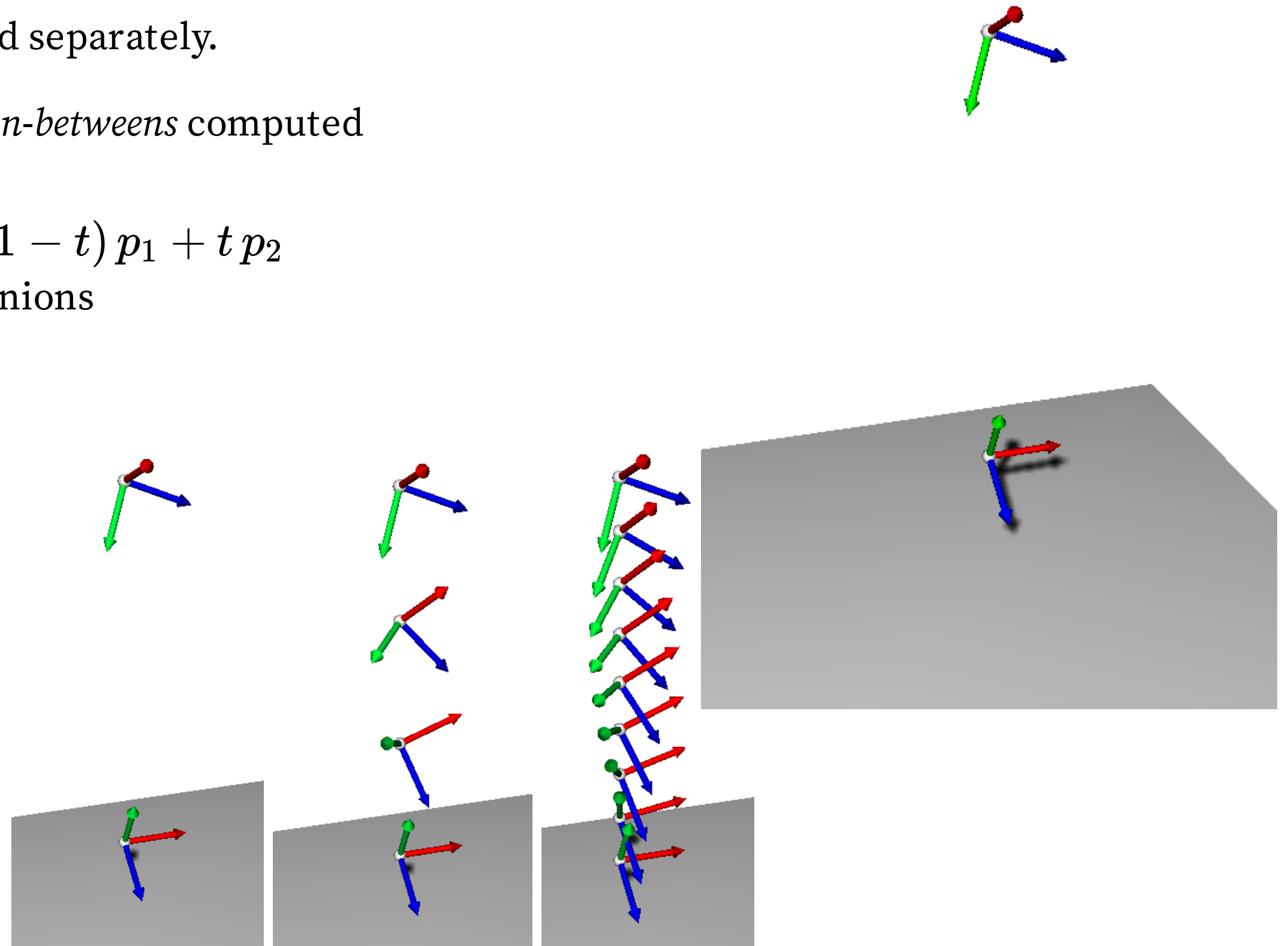
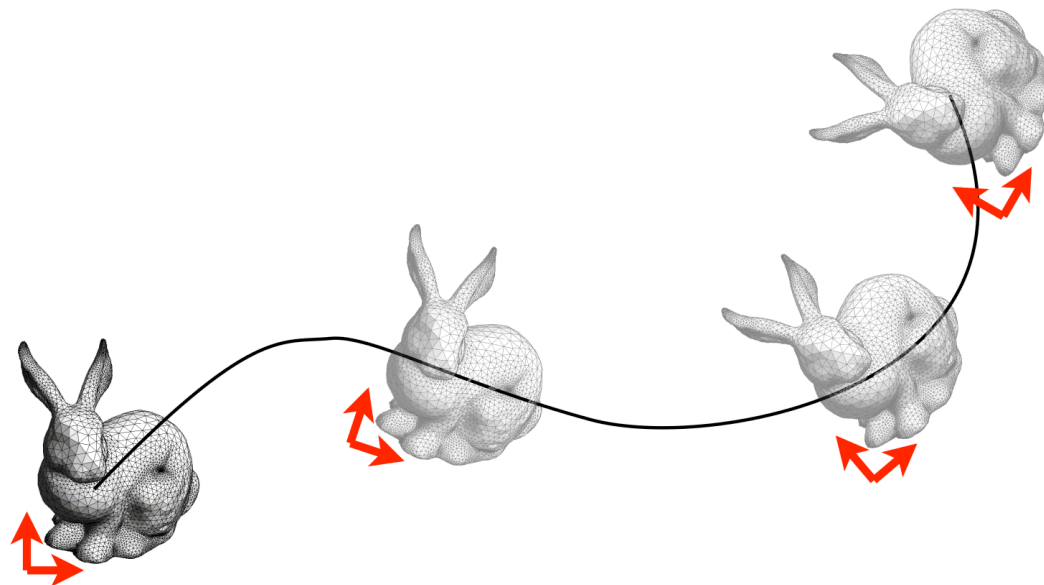
Interpolating rigid motion

3D objects have orientation r and position p .

⇒ Need to interpolate both, usually handled separately.

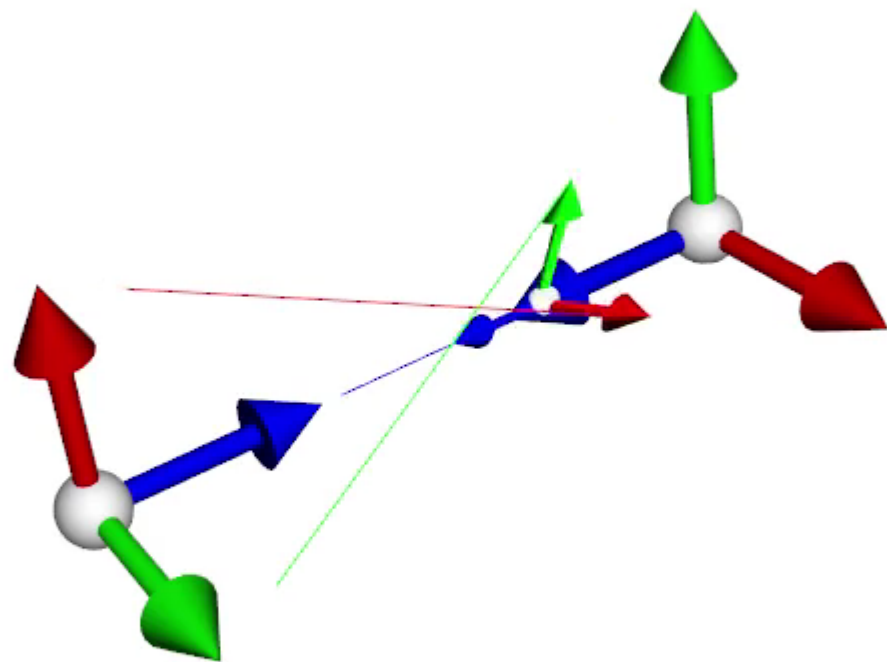
Given two key-positions (p_1, r_1) and (p_2, r_2) *in-betweens* computed at time $t \in [0, 1]$ as

- Linear interpolation of positions $p(t) = (1 - t)p_1 + tp_2$
- Interpolate rotation with SLERP on quaternions
 - Convert $(r_1, r_2) \rightarrow (q_1, q_2)$
 - Compute $q(t) = \text{SLERP}(q_1, q_2, t)$
 - Convert back $q(t) \rightarrow r(t)$

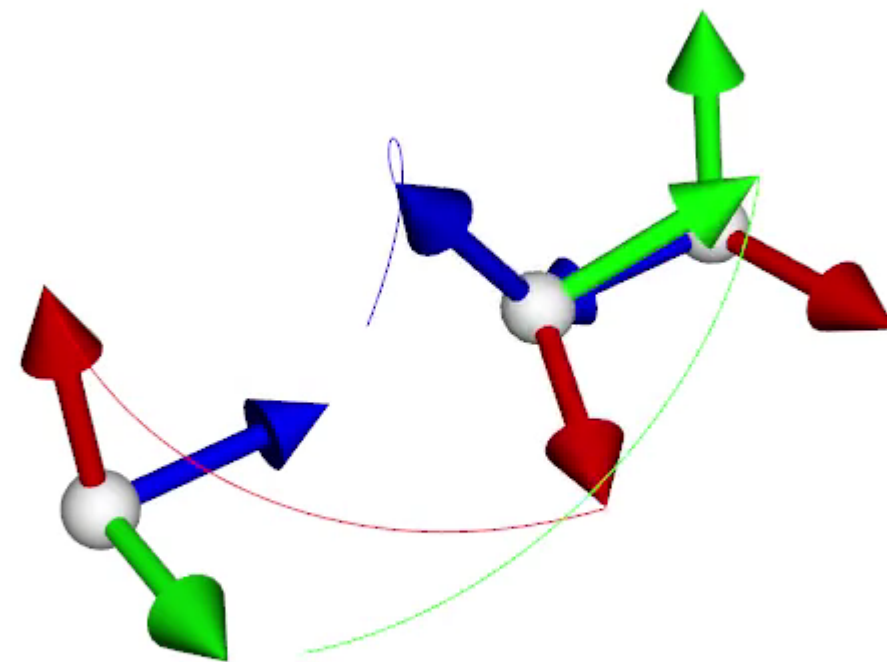


Interpolating rigid motion - Comparison

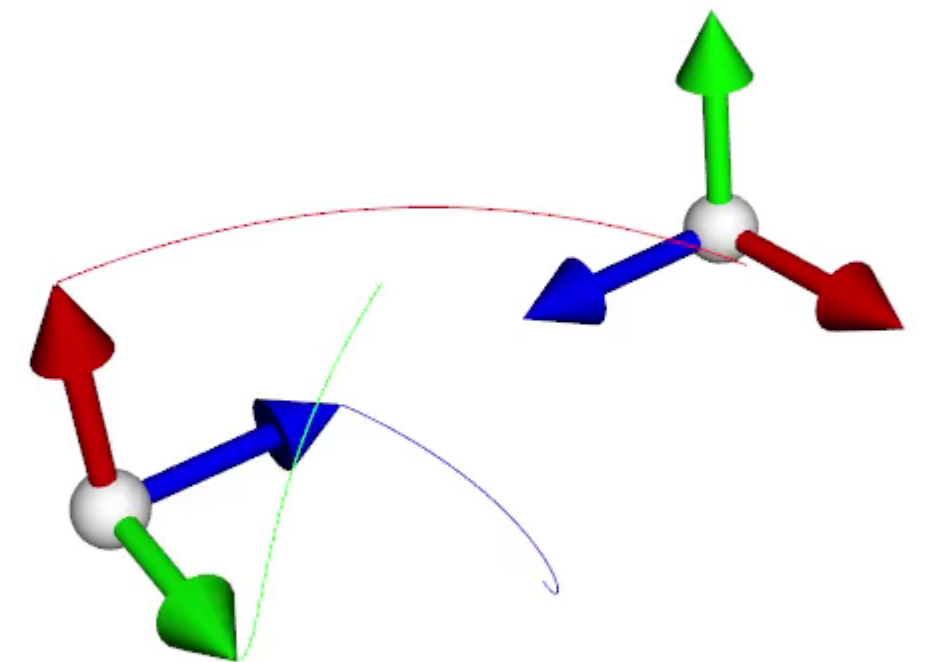
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Matrix interpolation



Euler angle interpolation



Quaternion interpolation