# MPRI 2.39 - Surface representation

# Representing surfaces

- **Meshes** *(saw in previous class)*
- Parametric Spline
- Subdivision
- Point set

# Mesh: pro cons

**Pros**

The simplest representation

The most general (can approximate any sampled surface)

The only natively handled by GPU (fast rendering)

**Cons**

$C^0$ only: The worst approximation for smooth surface.

$\Rightarrow$ High number of samples; complex modeling

# Representing surfaces

- Meshes
- **Parametric Spline**
- Subdivision
- Point set

# Objective of Parametric Spline

*Triangle: simplest linear element*

Parametric spline **objectives**

- Allow higher order element (less samples)
- Fully defined by positions (modeling purpose)
- $C^2$ smoothness (tangential + curvature continuity)

    *Allows smooth reflexion*

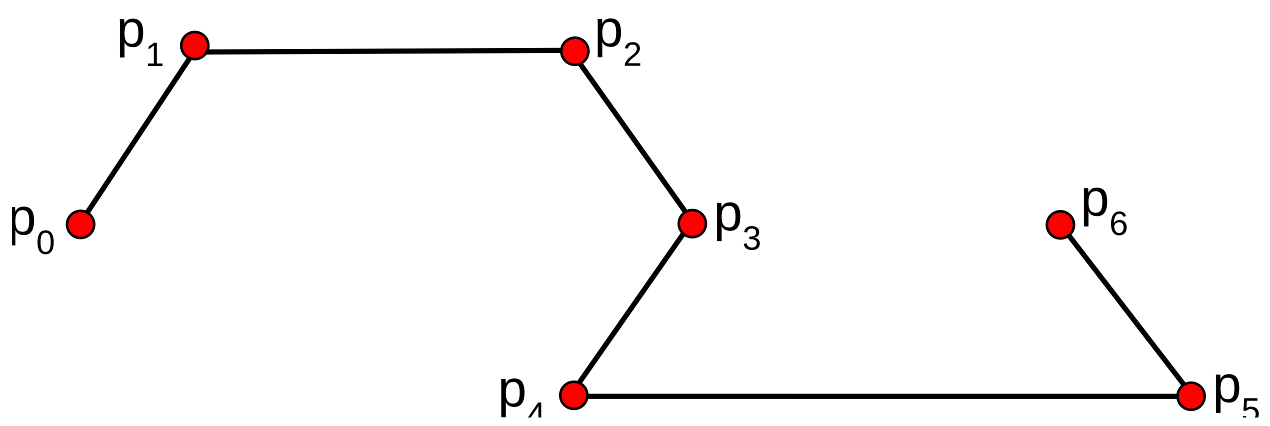
# Parametric Spline for 1D curves

General form: $c(s) = \sum_k b_k(s)\, p_k$

$b_k$: basis functions (smooth polynomial, minimal support)
$P_k$: control points (user defined)

**Straight segment**

$$c(s) = (1-s)\, p_0 + s\, p_1$$



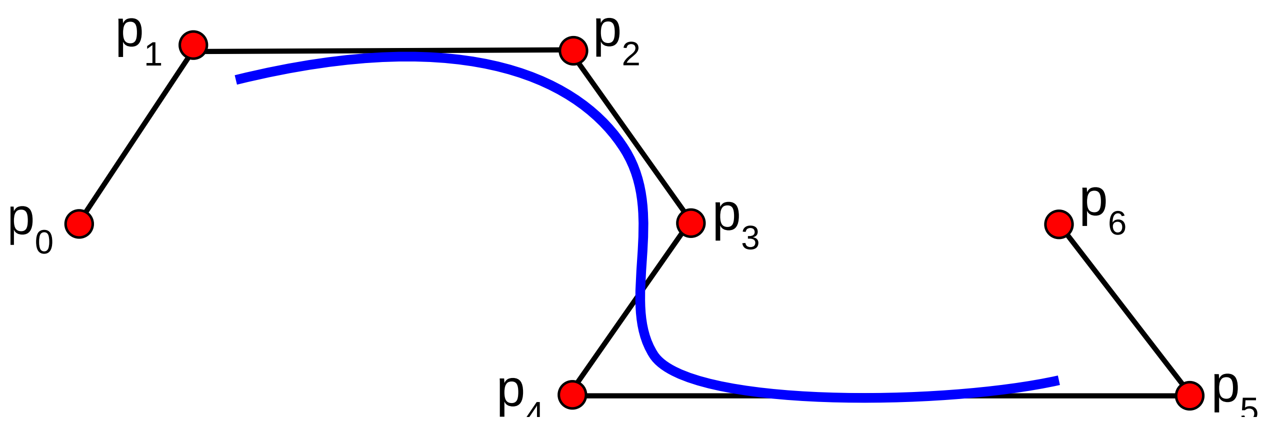

**Cubic uniform B-spline**

$$c(s) = \sum_{k=0}^{3} b_k(s)\, p_k$$

$$b_{i,k+1}(s) = \frac{s-t_i}{t_{i+k}-t_i}\, b_{i,k}(s) + \frac{t_{i+k+1}-s}{t_{i+k+1}-t_{i+1}}\, b_{i+1,k}(s)$$

$$b_{i,1}(s) = 1\ t_i \le s \le t_{i+1} \quad , \quad 0 \text{ otherwise}$$

$$\begin{cases} b_0(s) = (1-s)^3/6 \\ b_1(s) = (3s^3 - 6s^2 + 4)/6 \\ b_2(s) = (-3s^3 + 3s^2 + 3s + 1)/6 \\ b_3(s) = s^3/6 \end{cases}$$

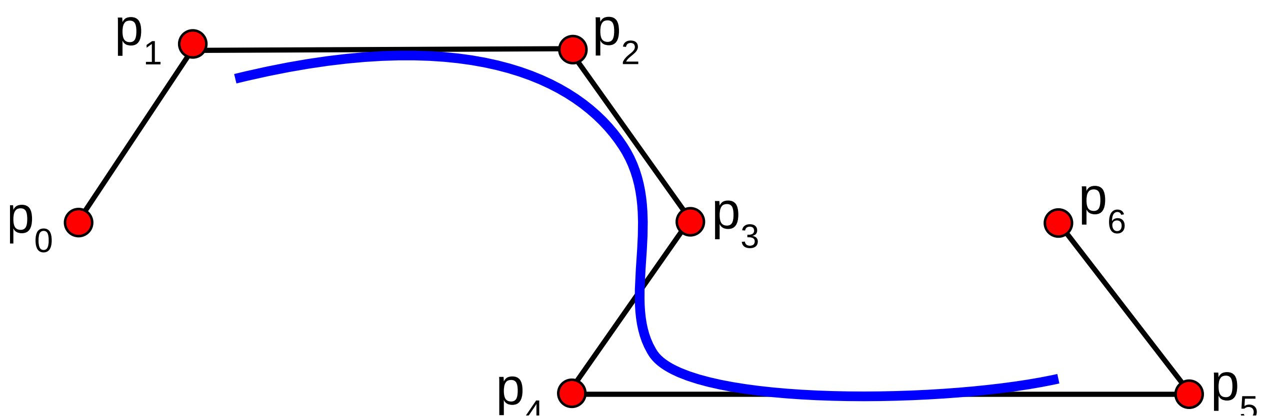


# Parametric Spline for Surfaces

*Standard approach:* Tensor product of polynomials

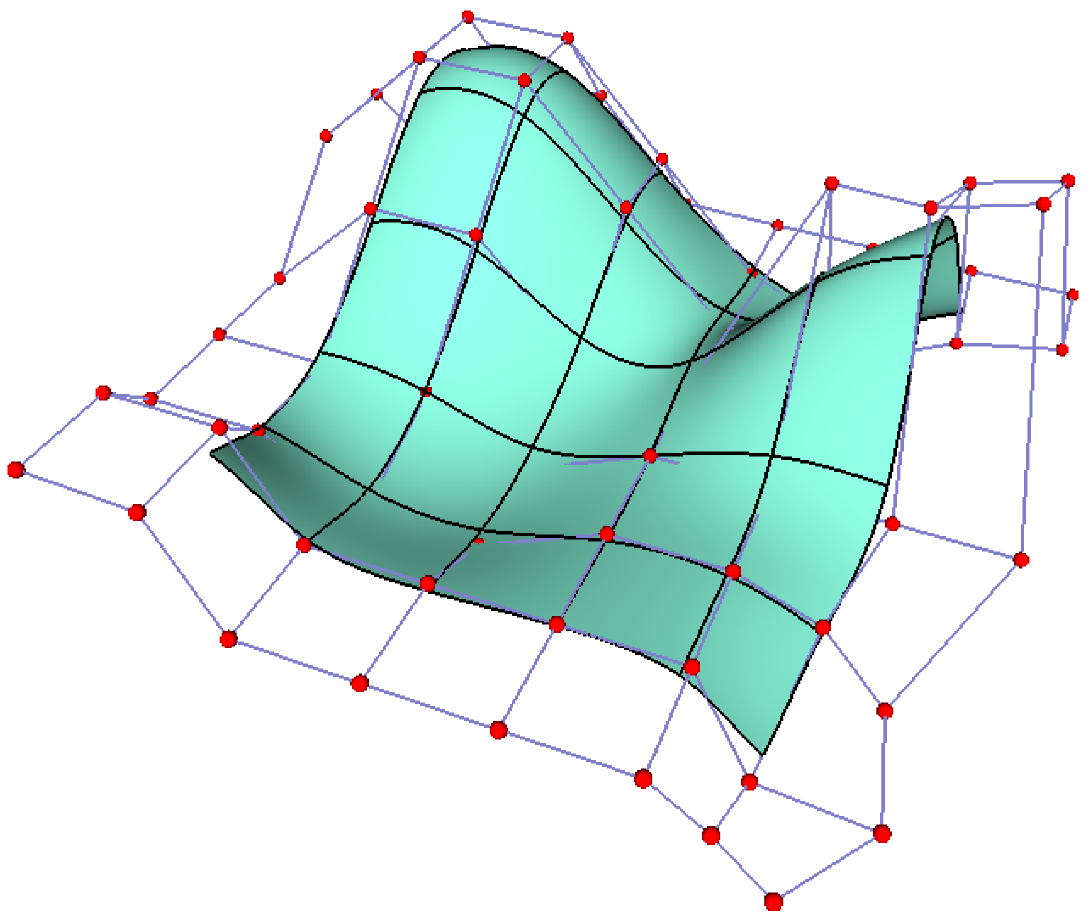
**Curve**

$$c(s) = \sum_k b_k(s)\, p_k$$




**Surface**

$$S(u,v) = \sum_i \sum_j b_i(u)\, b_k(v)\, p_{ij}$$



# Parametric Spline for Surfaces, matrix formulation

**Curve**

$$c(s) = \sum_k b_k(s)\; p_k$$

$$c(s) = \begin{pmatrix} s^3 & s^2 & s & 1 \end{pmatrix} \mathrm{M} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

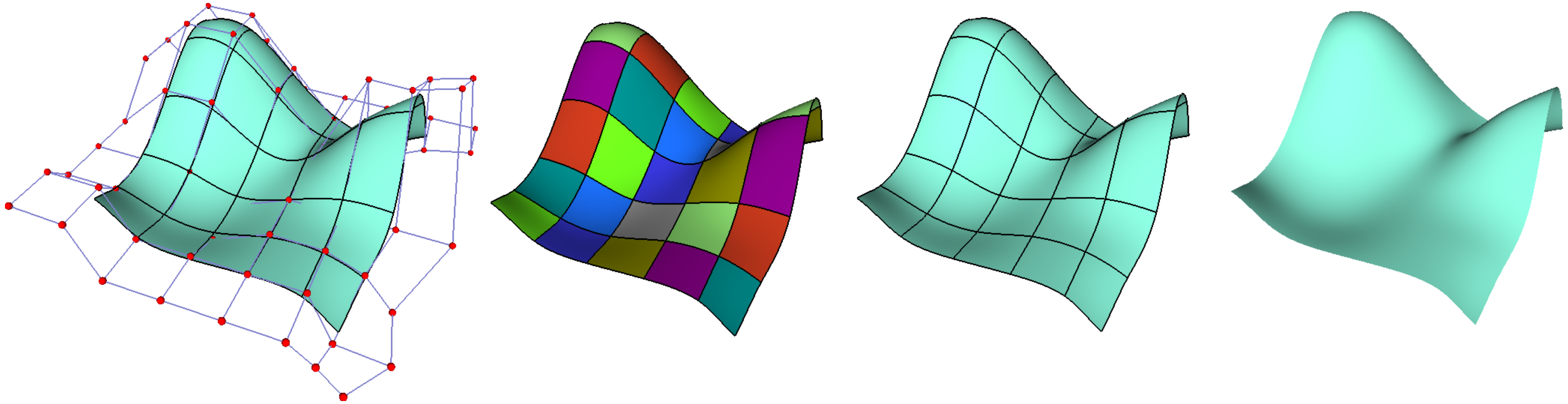
**Surface**

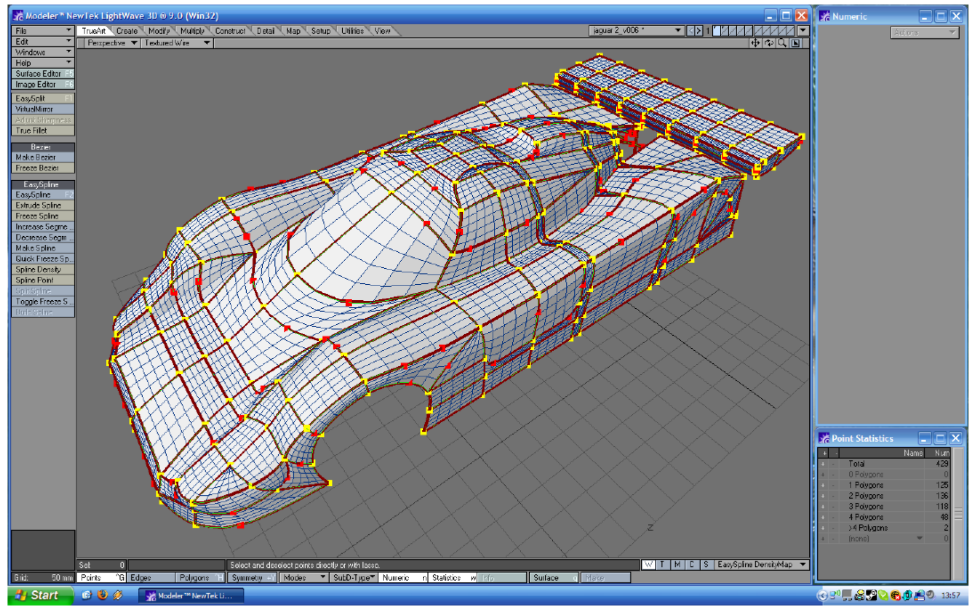
$$S(u,v) = \sum_i \sum_j b_i(u)\, b_k(v)\; p_{ij}$$

$$S(u,v) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \mathrm{M} \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{pmatrix} \mathrm{M}^t \begin{pmatrix} v^3 \\ v^2 \\ v \\ 1 \end{pmatrix}$$

$$\mathrm{M} = \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}$$ fully defined by the basis functions.

# Parametric Spline for 1D curves



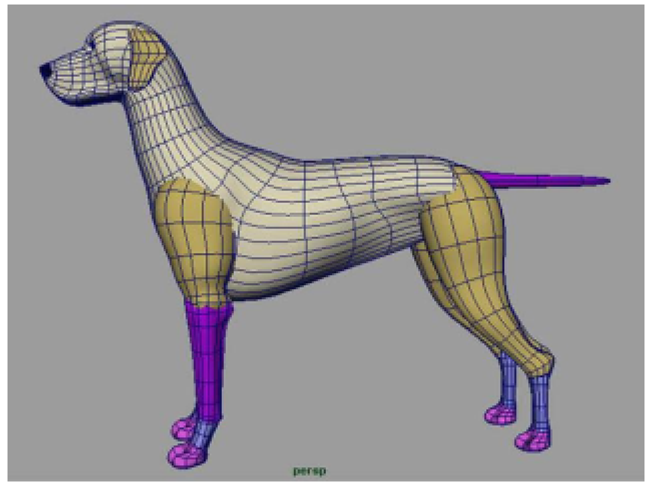


**Pro**

- $C^2$ smoothness
- Derivative/curvature computation

**Cons**

- Limited to $4 \times 4$ grid based patch topology



*How to merge surfaces, add a branch, etc*

# Representing surfaces

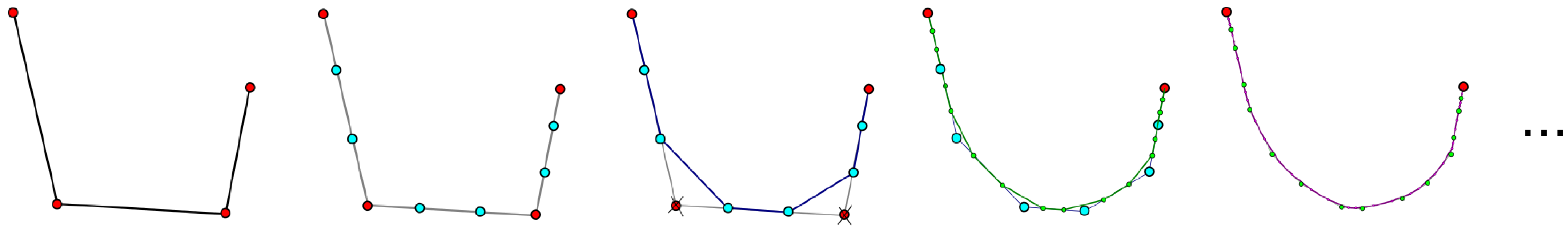
- Meshes
- Parametric Spline
- **Subdivision**
- Point set

# Objective of Subdivision surface

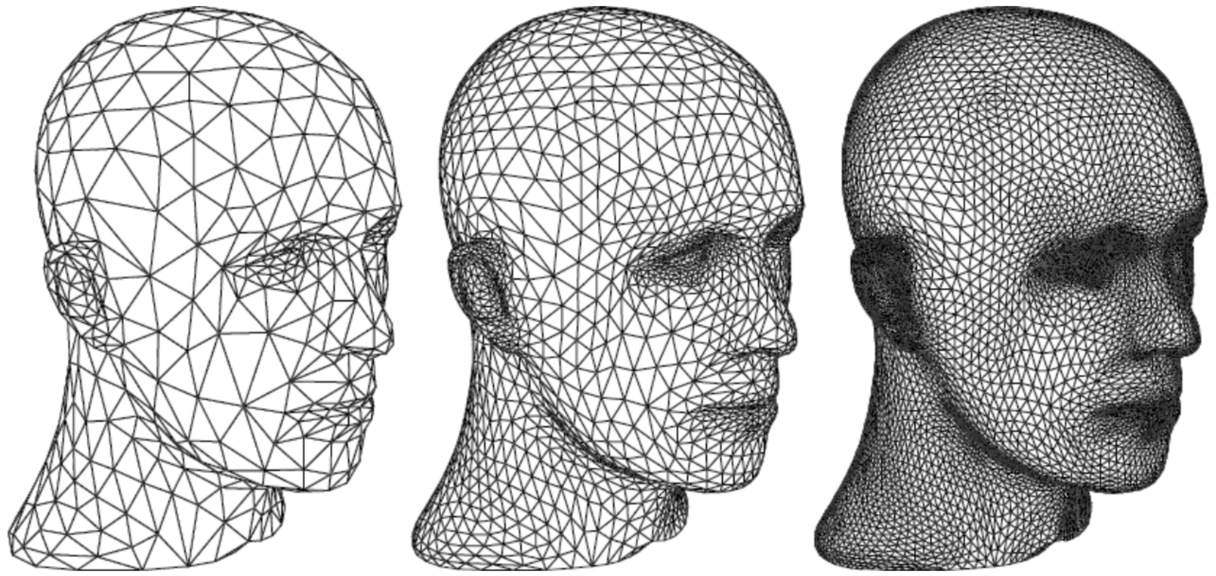
Keep advantages from
- **general topology** of meshes
- **smoothness** of parametric splines

$$\begin{pmatrix} p_{2k}^{n+1} \\ p_{2k+1}^{n+1} \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} p_k^{n+1} \\ p_{k+1}^{n+1} \end{pmatrix}$$

Example on **curves**



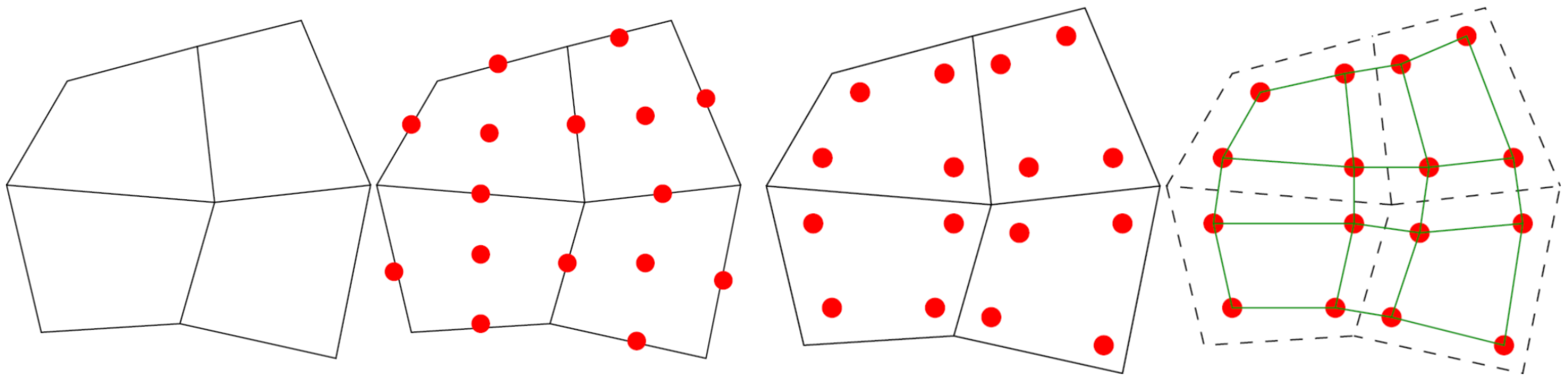

For **surfaces:** 2D subdivision mask

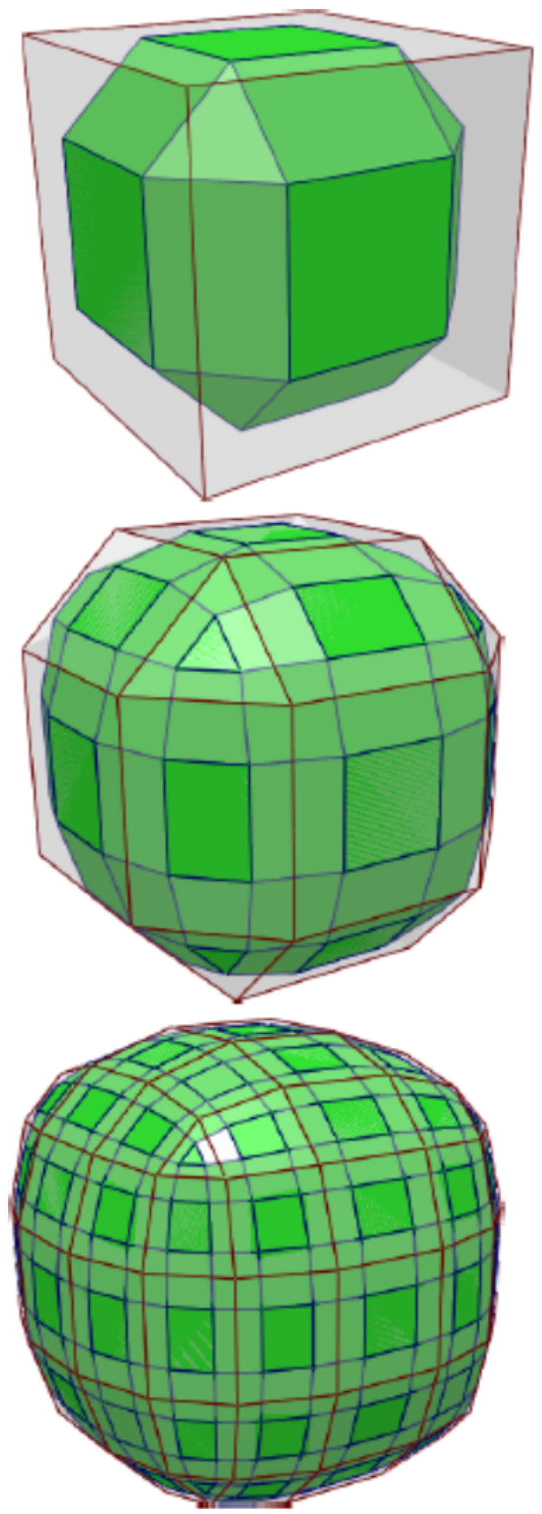


# Exemple of Subdivision Surface: Doo-Sabin

Given a face $(p_i)_i = [0, N-1]$

- Compute middle vertex $m_i = (p_i + p_{i+1})/2$
- Compute barycenter of the face $b = 1/N \sum_{i=0}^{N-1} p_i$
- New vertices are $n_i = (p_i + m_i + m_{i-1} + b)/4$





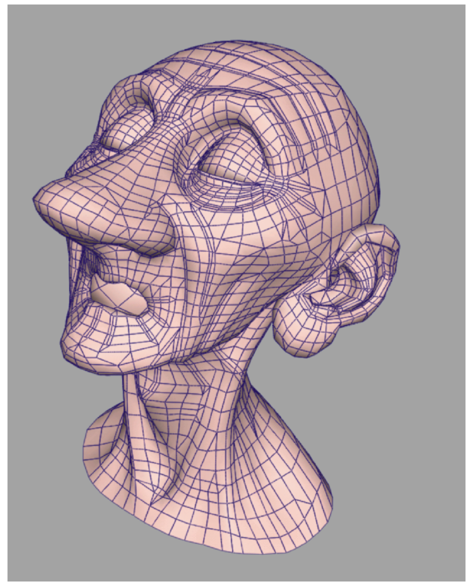
# Subdivision surfaces

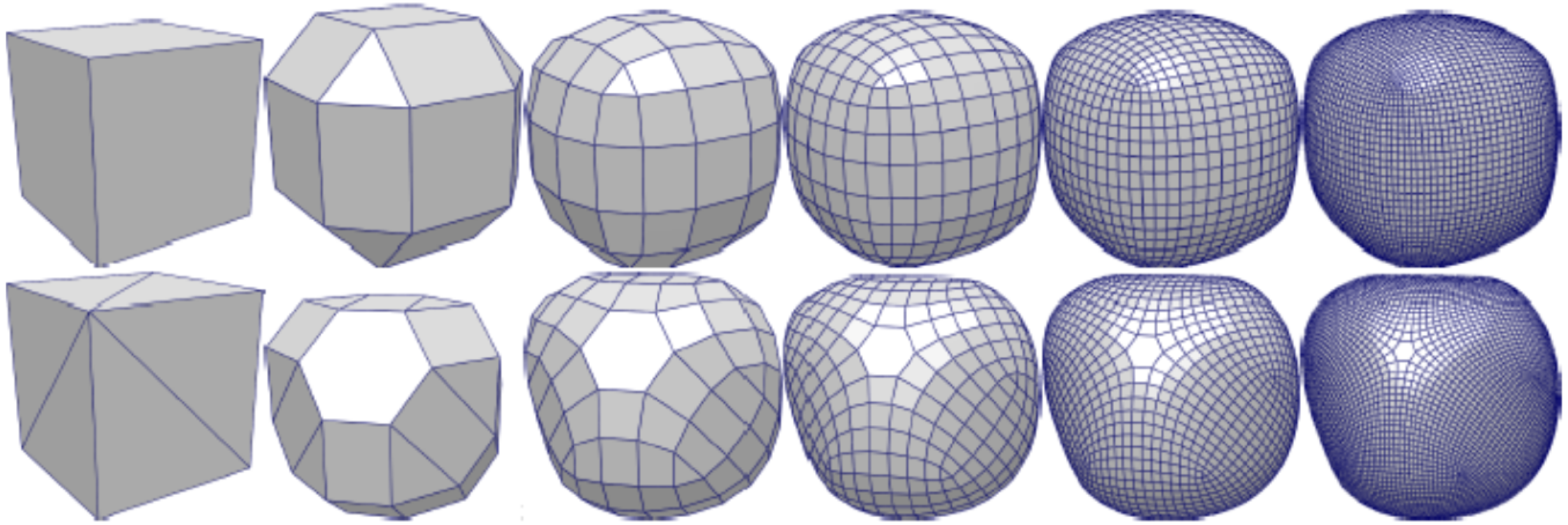
**Pros**

- Arbitrary input mesh topology
- Guaranteed convergence to $C^1$ or $C^2$ smooth surface.

**Cons**

- Final shape is hard to predict (depends on scheme, connectivity)
- No direct access to differential values





# Representing surfaces

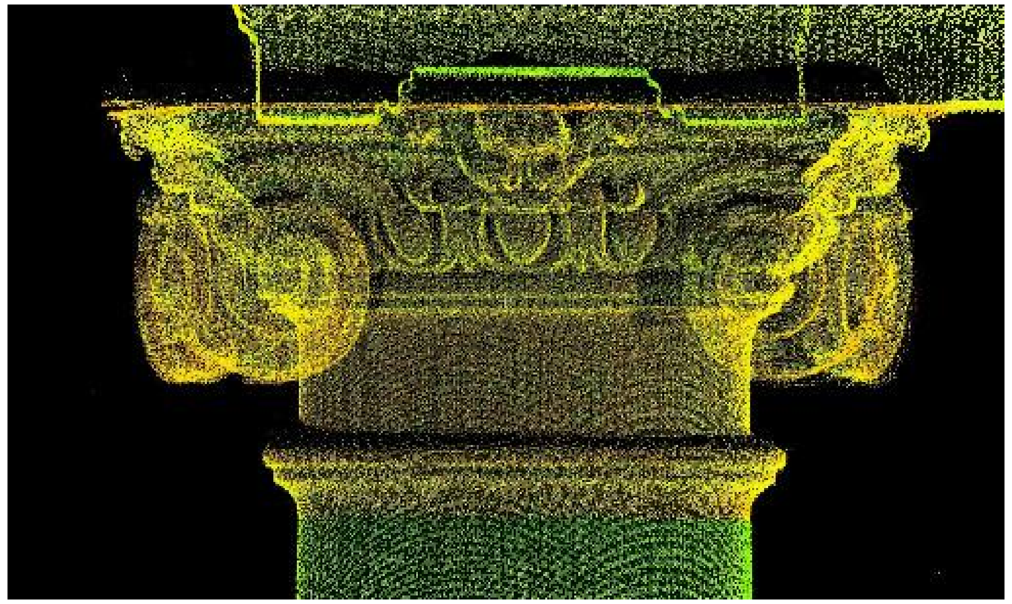
- Meshes
- Parametric Spline
- Subdivision
- **Point set**

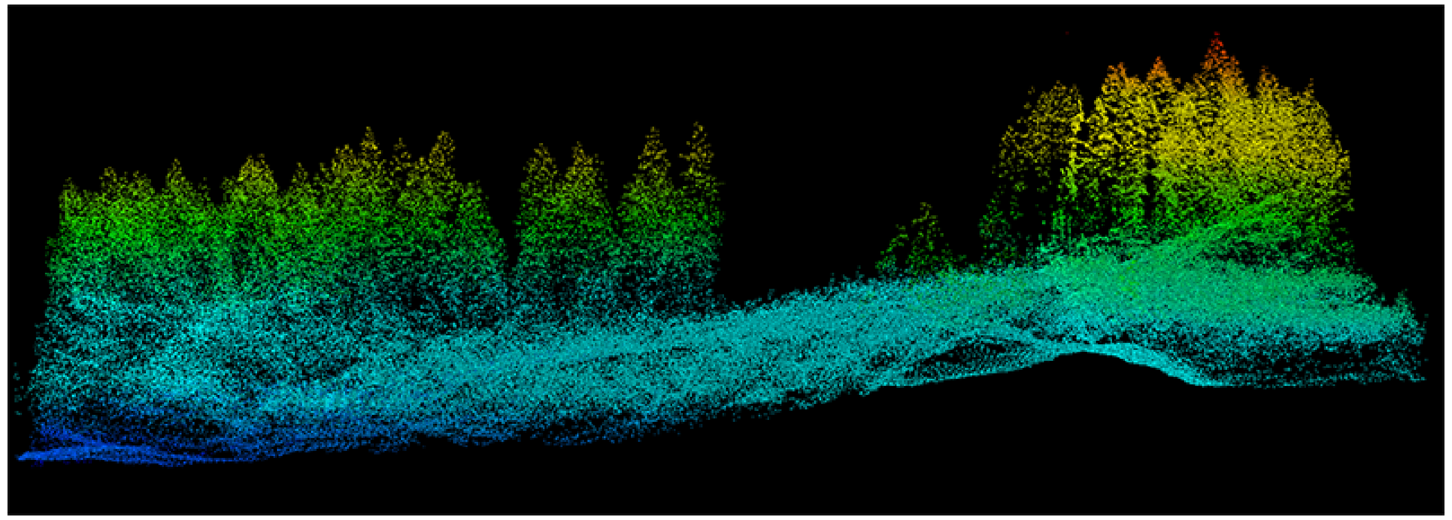
# Point set

Raw 3D point sets are widely avaiable from 3D scanners

- No explicit connectivity
- Huge amount of points ($10^{7-10}$)

Idea: Represent entire surfaces from point sets only





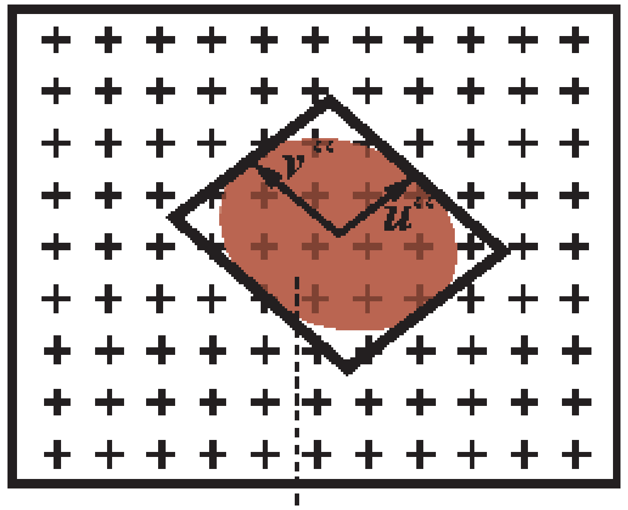
# Point set rendering

Surface looking from point sets only requires specific rendering

=> **Surface splatting**

- Cover surface by **surfels** - *oriented disks tangent to surface*
- Render them in image space as ellipsoids


*In 3D space*




*Render in image space*

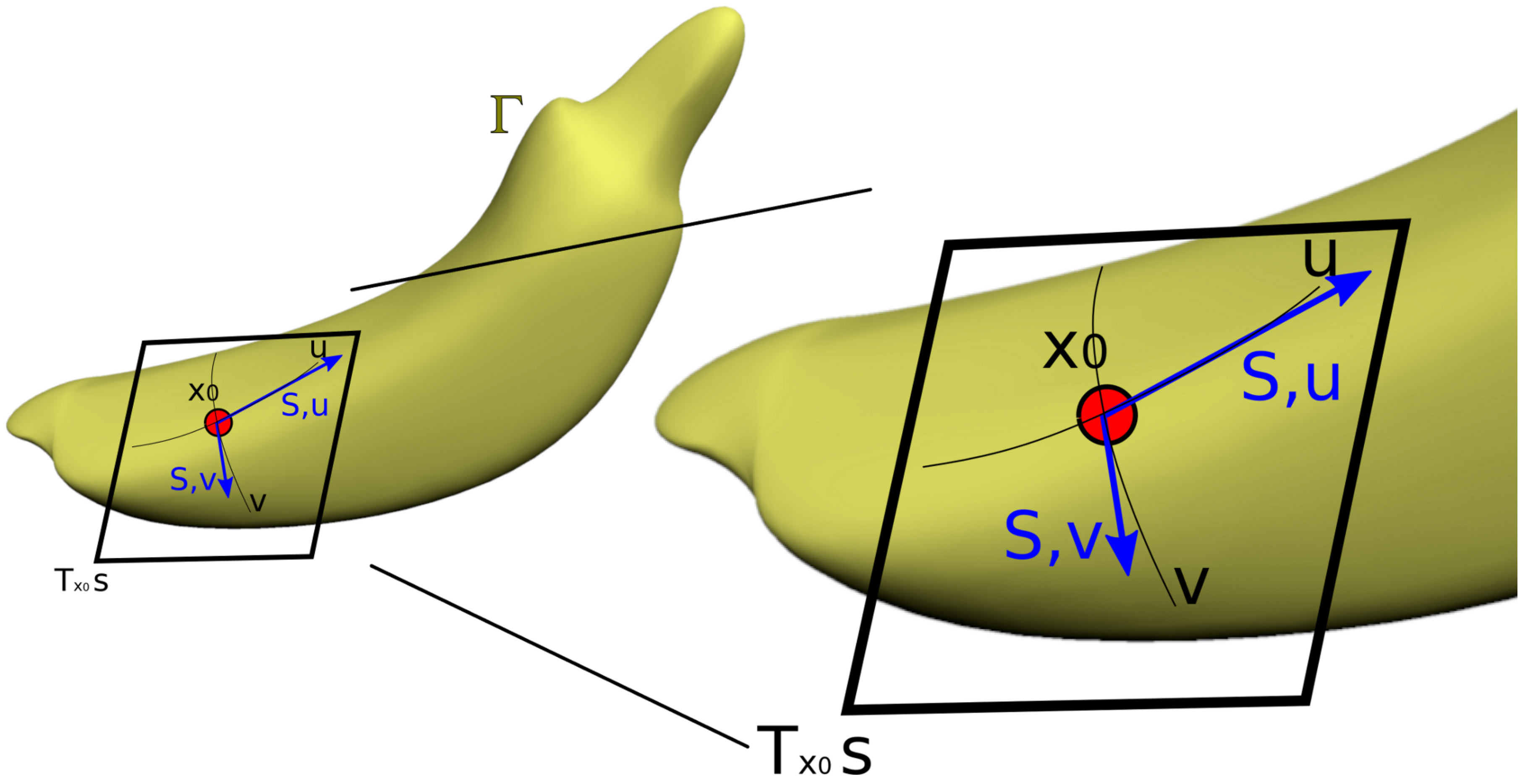
# MPRI 2.39 - Smooth Parametric Surfaces and Differential Geometry

# Parametric smoothness

Let $S$ be mapping of a surface $S : \begin{cases} \mathcal{D} \subset \mathbb{R}^2 & \to \mathbb{R}^3 \\ (u,v) & \mapsto S(u,v) \end{cases}$

- $S$ is $\mathcal{C}^1$ if $S_{,u}$ and $S_{,v}$ are defined and continuous
- $S$ is $\mathcal{C}^2$ if $S_{,uu}$, $S_{,vv}$ and $S_{,uv}$ are defined and continuous

$S_{,u} = \frac{\partial S}{\partial u}$, $S_{,uv} = \frac{\partial^2 S}{\partial u \partial v}$



# Geometric smoothness

Let $\Gamma$ be the image of the mapping $S$.

- $\Gamma$ is $G^1$ it has a tangent plane in every position
- $\Gamma$ is $G^2$ it has curvature in every position

**Note**

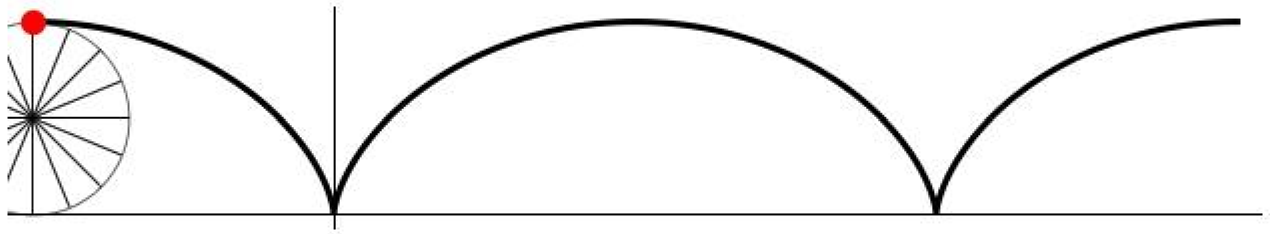
- $S \subset \mathcal{C}^k \neq \Gamma \subset G^k$
- $G^2$ is required for smooth reflexion within the surface.

# Parametric vs Geometric smoothness example

Are these functions $\mathcal{C}^1, \mathcal{C}^2, G^1, G^2$ ?

$$f(t) = \left| \begin{array}{ll} (t,t) & , \quad t \in [-1,0[ \\ (t/2,t/2) & , \quad t \in [0,2] \end{array} \right.$$

$$h(t) : \begin{cases} x(t) = R(t - \sin(t)) \\ y(t) = R(1 - \cos(t)) \end{cases}$$

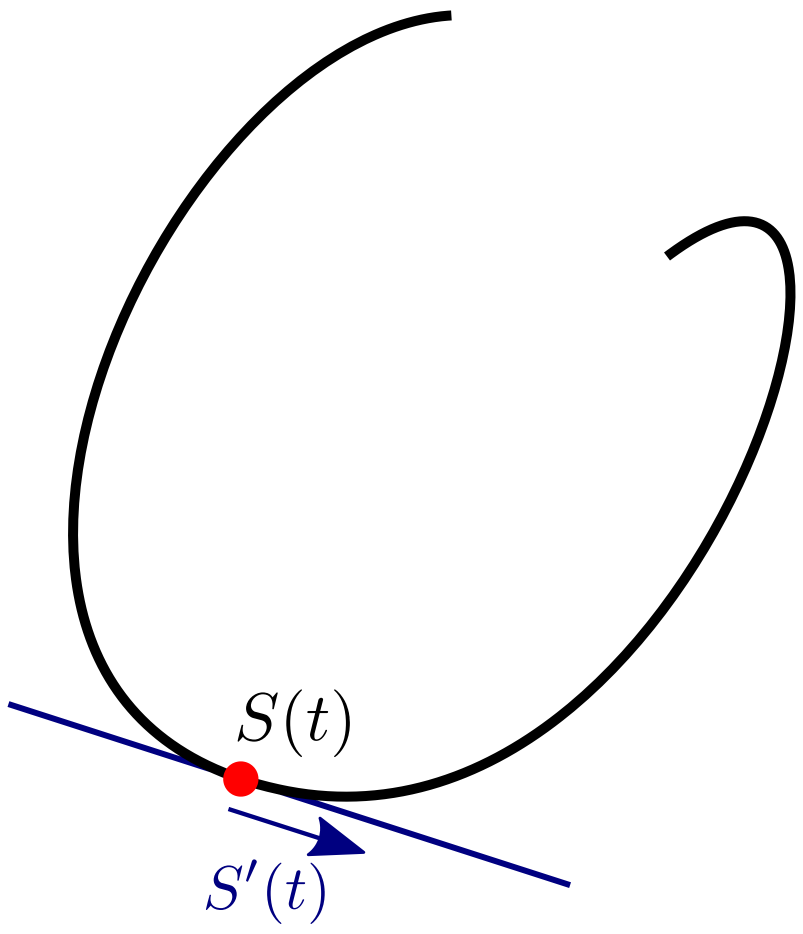


# Reminder about parametric curve

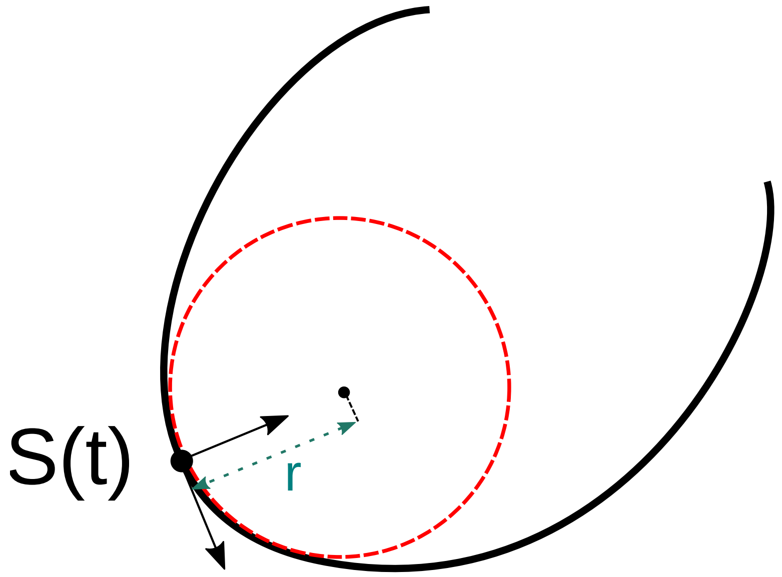
Curve $S(t)$, $S : \begin{cases} \mathbb{R} \to \mathbb{R}^3 \\ t \to (S_x(t), S_y(t), S_z(t)) \end{cases}$

- Tangent vector at position $S(t)$: $T(t) = S'(t)/\|S'(t)\|$

- Length of the curve $L = \displaystyle\int_t \|S'(t)\| \, \mathrm{d}t$



# Curvature of a curve




**Geometric interpretation**

- Radius of curvature $r$ : radius of the osculating circle.
- Curvature $\kappa = 1/r$.

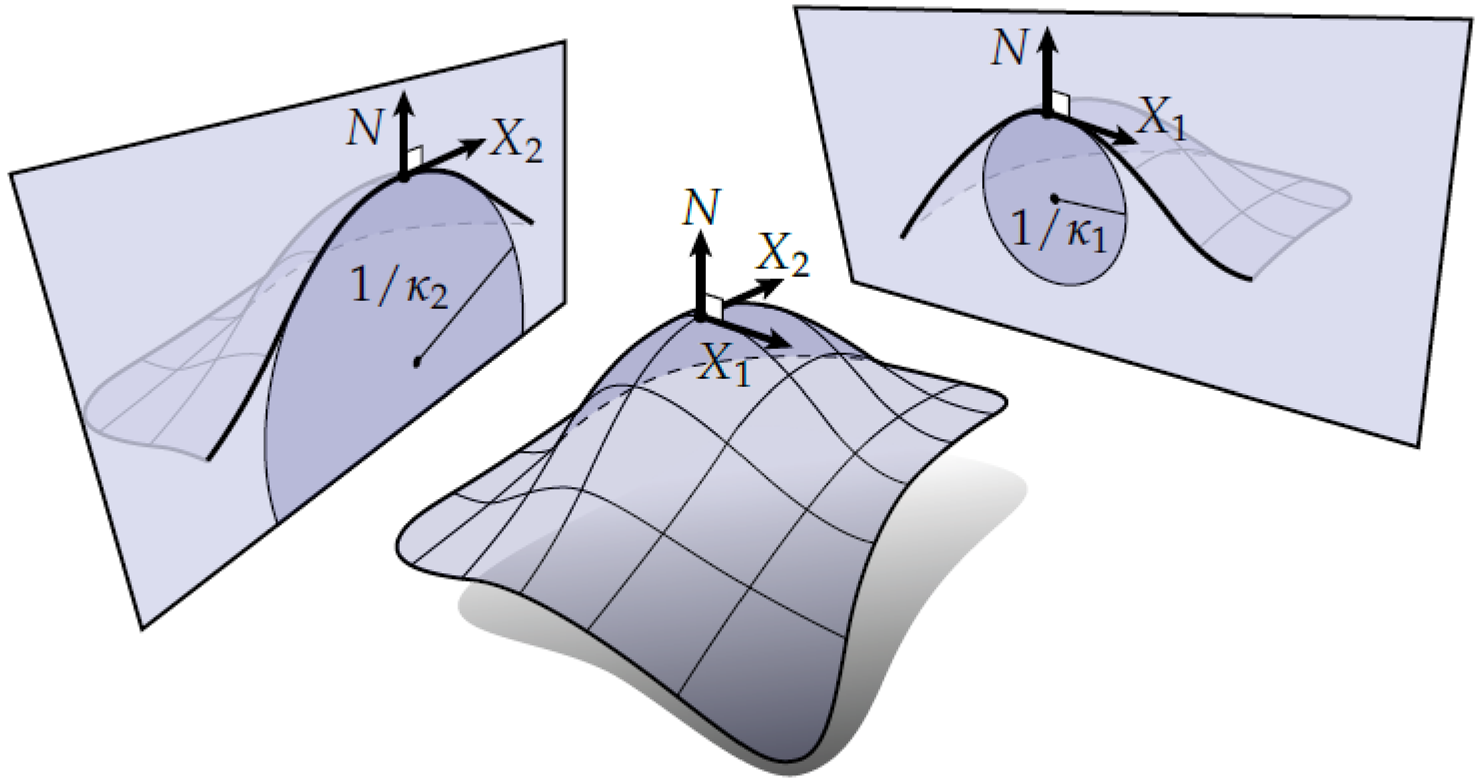
**Definition** Consider the curvilinear parameterization $\mu$ such that $\|S'(\mu)\| = 1$.

- $|\kappa(\mu)| = \left\| \dfrac{\partial T}{\partial \mu}(\mu) \right\|$ : *rate of change of the tangent vector*
- $\kappa$ is considered positive/negative with respect to the direction of the normal vector.

**Computational definition**

$$|\kappa(t)| = \frac{\|S'(t) \times S''(t)\|}{\|S'(t)\|^3}$$

# Curvature of a surface : intuition




Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus. ACM SIGGRAPH 2013

# Tangent Plane and Normal to a Surface

- Parametric Surface $S(u,v) = (S_x(u,v), S_y(u,v), S_z(u,v))$ with image $\Gamma$.

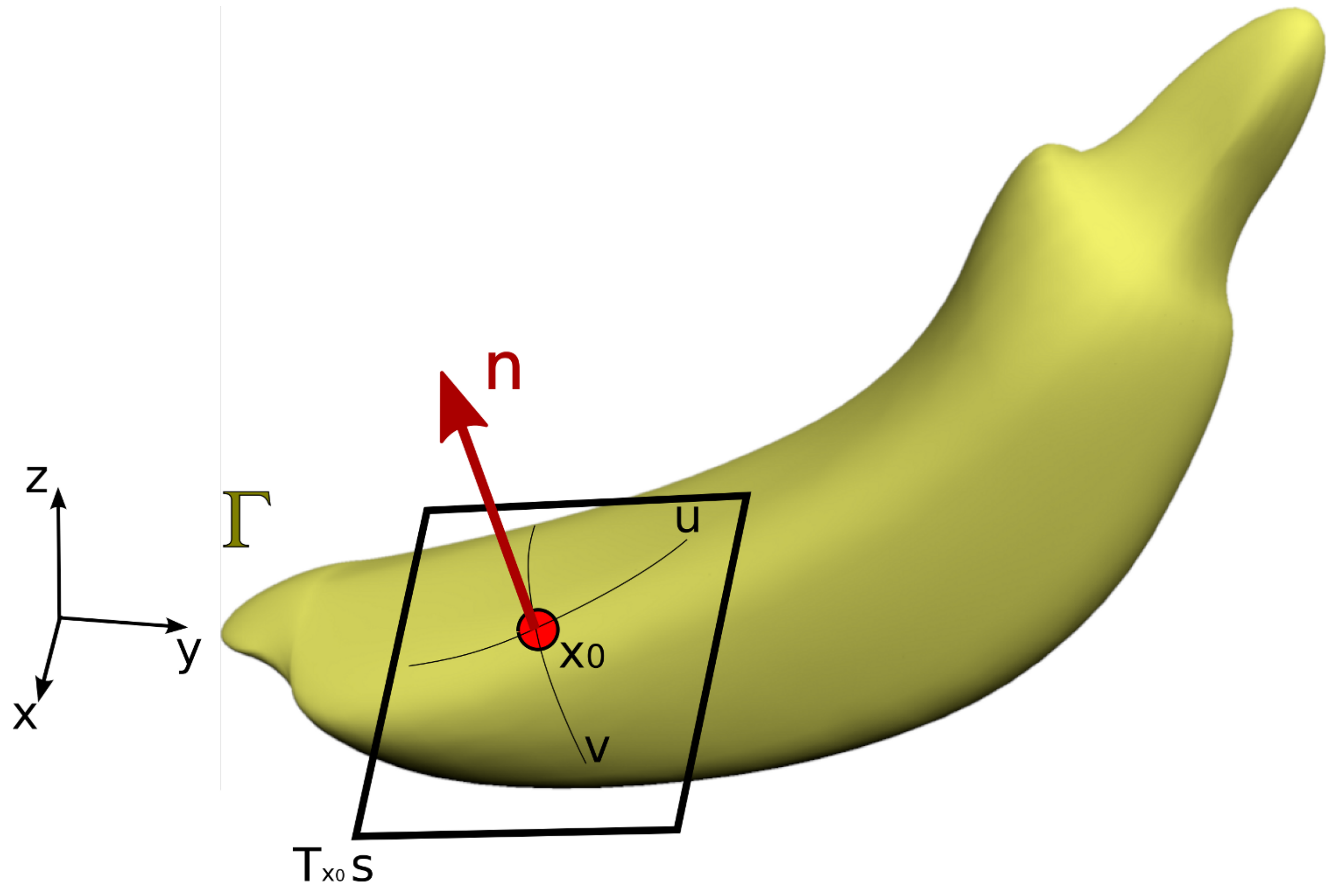
- Tangent space of $S$ at $p = S(u,v)$

$$T_p(S) = \{\ S_{,u}(u,v)\, h_u + S_{,v}(u,v)\, h_v \mid (h_u, h_v) \in \mathbb{R}^2\ \}$$

- Unit normal at parameter $(u,v)$: $n(u,v) = (S_{,u} \times S_{,v}) / \|S_{,u} \times S_{,v}\| \in \mathbb{S}^2$

- The Gauss map $N$

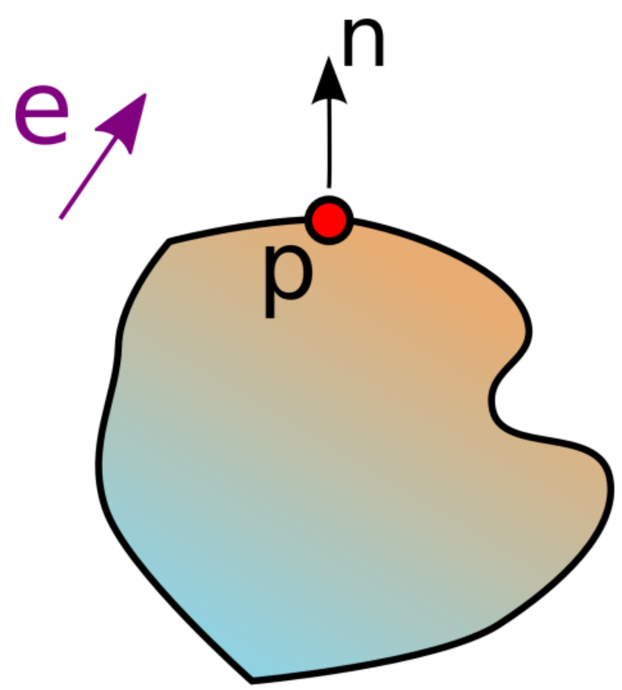
$$N(p) = \begin{cases} \Gamma & \to & \mathbb{S}^2 \\ p = S(u,v) & \to & N(p) = n(u,v) \end{cases}$$



# Integral properties

- Area $A$ of $\Gamma$ : $A = \iint_{(u,v)\in\mathcal{D}} \|S_{,u} \times S_{,v}\| \,\mathrm{d}u\,\mathrm{d}v$

- Volume $V$ defined by $\Gamma$ : $\forall e \in \mathbb{S}^2$ , $V = \iint_{(u,v)\in\mathcal{D}} < S(u,v), e > < n(u,v), e > \,\mathrm{d}u\,\mathrm{d}v$




- *Definition of the Volume* $V = \iiint_{\Omega} \mathrm{d}\Omega$
- *Green-Ostrogradsky / Divergence theorem* $\iiint_{\Omega} \mathrm{div}(f)\,\mathrm{d}\Omega = \iint_{S} < f, n > \,\mathrm{d}S$
- *We set* $f(p) =< p, e > \; e$ *for any* $e \in \mathbb{S}^2$

  $(\mathrm{div}(f) = \|e\|^2 = 1)$

$\Rightarrow V = \iiint_{\Omega} 1\mathrm{d}\Omega = \iint_{S} << S(p), e > \; e, n > \mathrm{d}S = \iint_{S} < S(p), e >< n, e > \mathrm{d}S$

# First fundamental form

- 2D curve $C \subset \mathcal{D}$

  $C(t) = (C_x(t), C_y(t)$

- Length of $C$ $L = \int_t \left(C'^T(t)\, C'(t)\right)^{1/2} \mathrm{d}t$

- 3D curve $C_s = S \circ C$

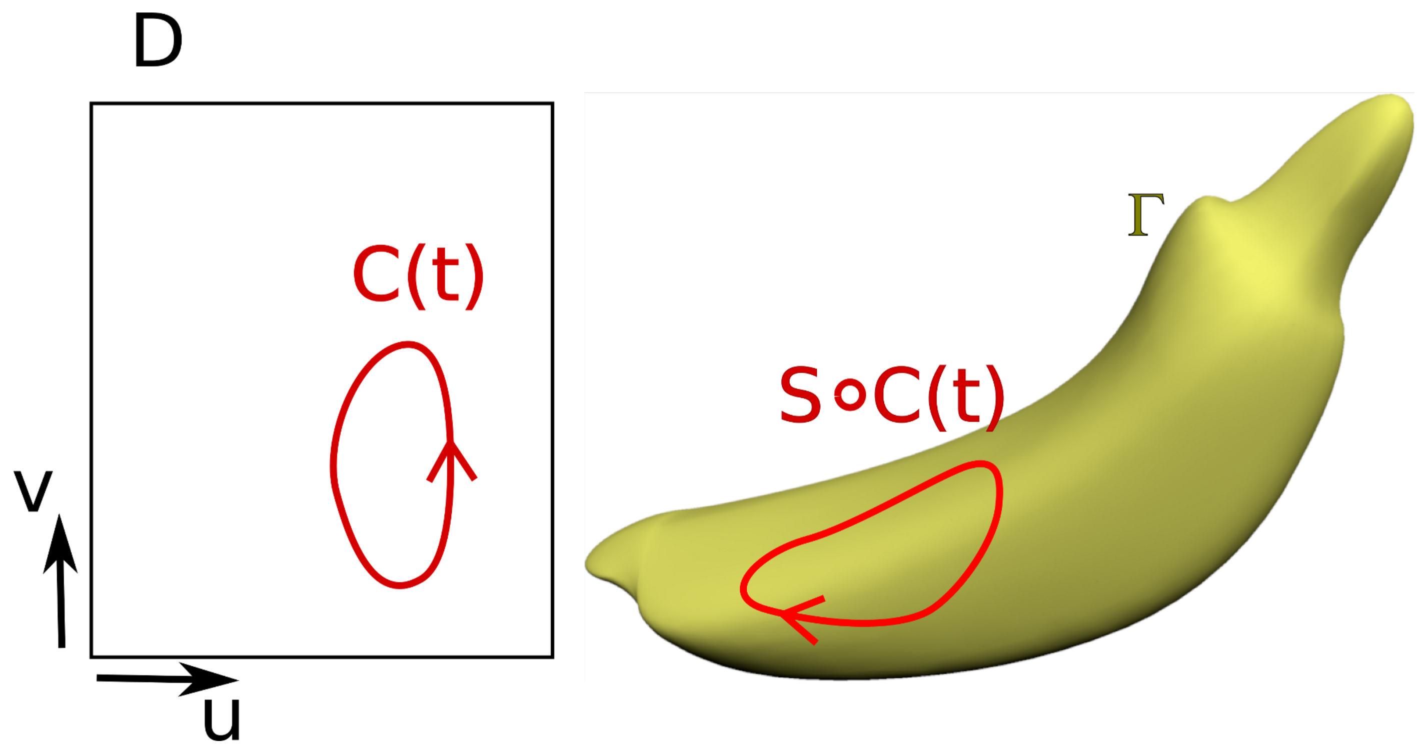
  $C_s(t) = \left(S_x(C_x(t), C_y(t)),\ S_y(C_x(t), C_y(t)),\ S_z(C_x(t), C_y(t)\right)$

- Length of 3D curve $L_s$

  $$L_s = \int_t \left(((S \circ C)'^T(t)\, (S \circ C)'(t)\right)^{1/2} \mathrm{d}t$$

  $$L_s = \int_t \left(C'^T(t)\ \mathrm{I}_S(t)\ C'(t)\right)^{1/2} \mathrm{d}t$$

  $\mathrm{I}_S(t)$ : first fundamental form



# First fundamental form derivation

- $L_s = \int_t \left((S \circ C)'^T(t)\,(S \circ C)'(t)\right)^{1/2}\,\mathrm{d}t$

- $(S \circ C)' = C'_x\,(S_{,u} \circ C) + C'_y\,(S_{,v} \circ C)$
- $(S \circ C)' = (S_{,u}\ S_{,v}) \begin{pmatrix} C'_x \\ C'_y \end{pmatrix} = \partial S^T\, C'$

- $(S \circ C)'^T\,(S \circ C)' = (C'^T\,\partial S)\,(\partial S^T\,C') = C'^T\,\mathrm{I}_s\,C'$
  $\mathrm{I}_s = \partial S\,\partial S^T$

# First fundamental form properties

$\mathrm{I}_S$ : first fundamental form / metric tensor

$$\mathrm{I}_S = \begin{pmatrix} S_{,u}^2 & < S_{,u}, S_{,v} > \\ < S_{,u}, S_{,v} > & S_{,v}^2 \end{pmatrix}$$

$\sqrt{\det(\mathrm{I}_S)}$ : Local change of area

$$\text{Area of } \Gamma = \iint_{(u,v)\in\mathcal{D}} \sqrt{\det(\mathrm{I}_S)}\, \mathrm{d}u\, \mathrm{d}v$$

# Second fundamental form

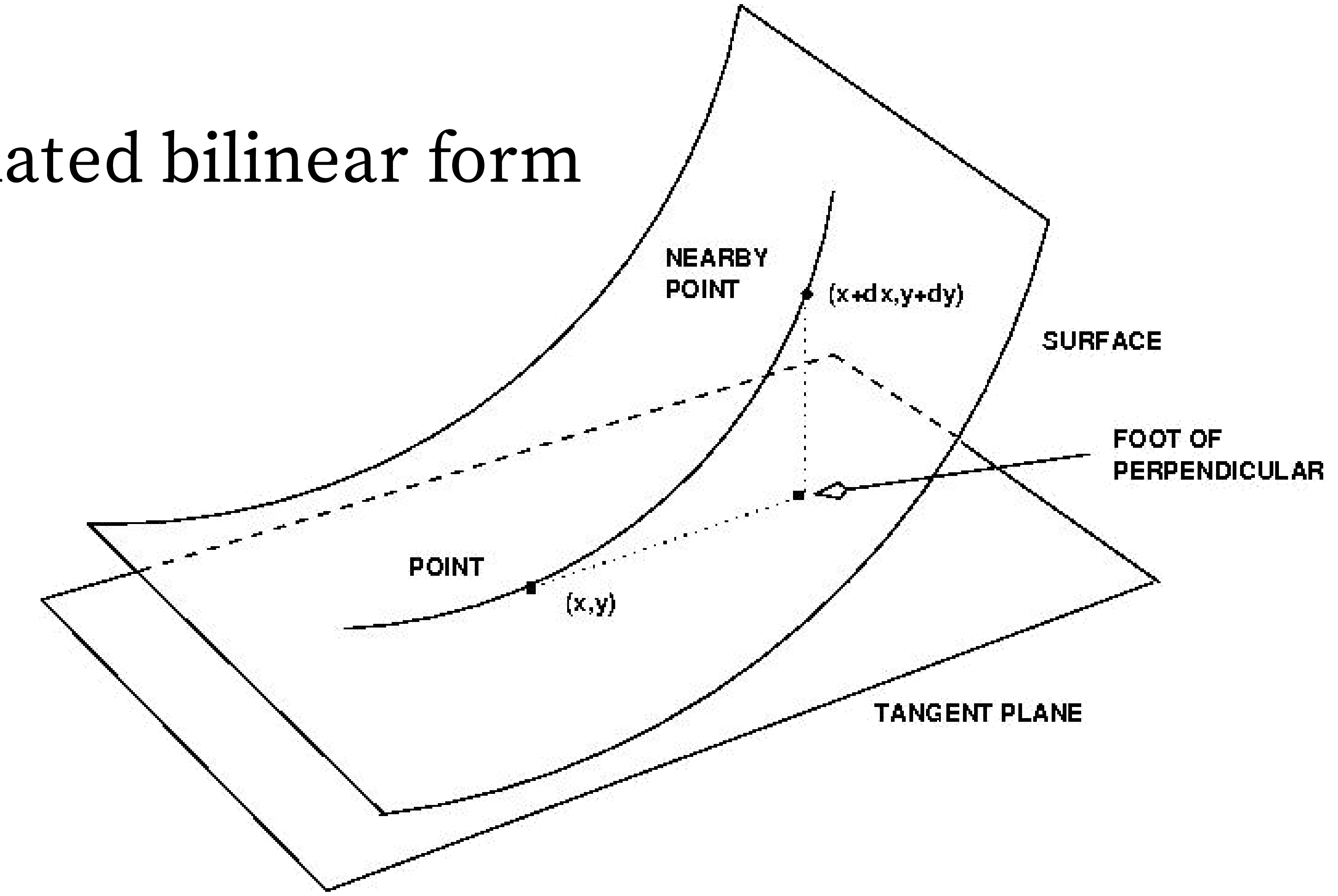
Consider $S$ to be locally defined in the local tangent plane.

The Taylor expansion of $S$ along the normal direction, at the second order is

$$z(u,v) = \frac{1}{2} < S_{,uu}, n > \,\mathrm{d}u^2 + \frac{1}{2} < S_{vv}, n > \,\mathrm{d}v^2 + < S_{,uv}, n > \,\mathrm{d}u\,\mathrm{d}v$$

We call the second fundamental form the associated bilinear form

$$\mathrm{II}_S = \begin{pmatrix} < S_{,uu}, n > & < S_{,uv}, n > \\ < S_{,uv}, n > & < S_{,vv}, n > \end{pmatrix}$$



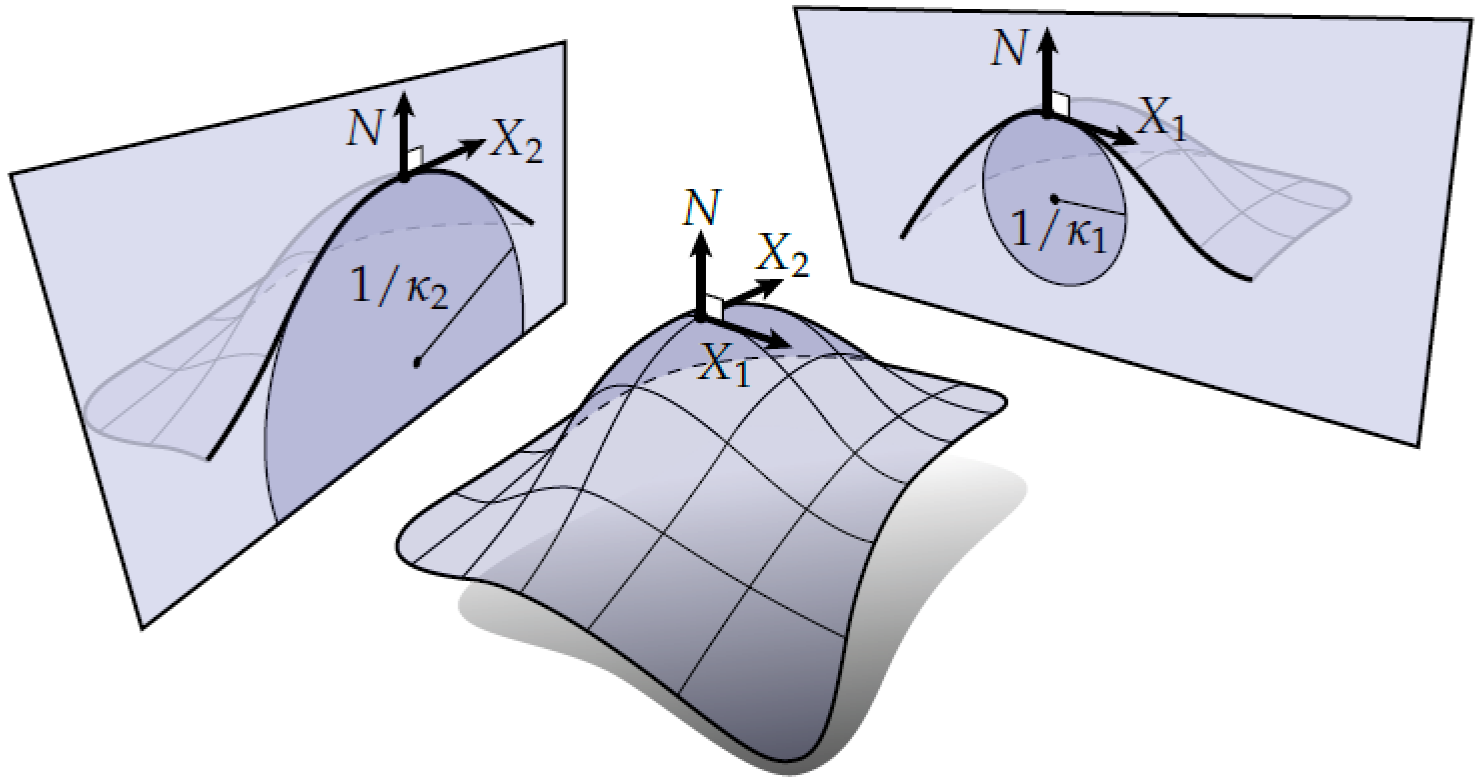
# Weingarten map and Curvature

- $\mathrm{W}_S = \mathrm{II}_s\, \mathrm{I}_S^{-1}$ is the Weingarten matrix (or shape operator - differential of the Gauss map).

- $\mathrm{W}_S$ is diagonalizable and has real eigenvalues. $\mathrm{W}_S = V^T \Lambda V$

$\Lambda = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$, $(\kappa_1, \kappa_2)$: two principal curvatures

Eigenvectors are the principal directions of curvature (in the local tangent plane).



# Local surfaces types

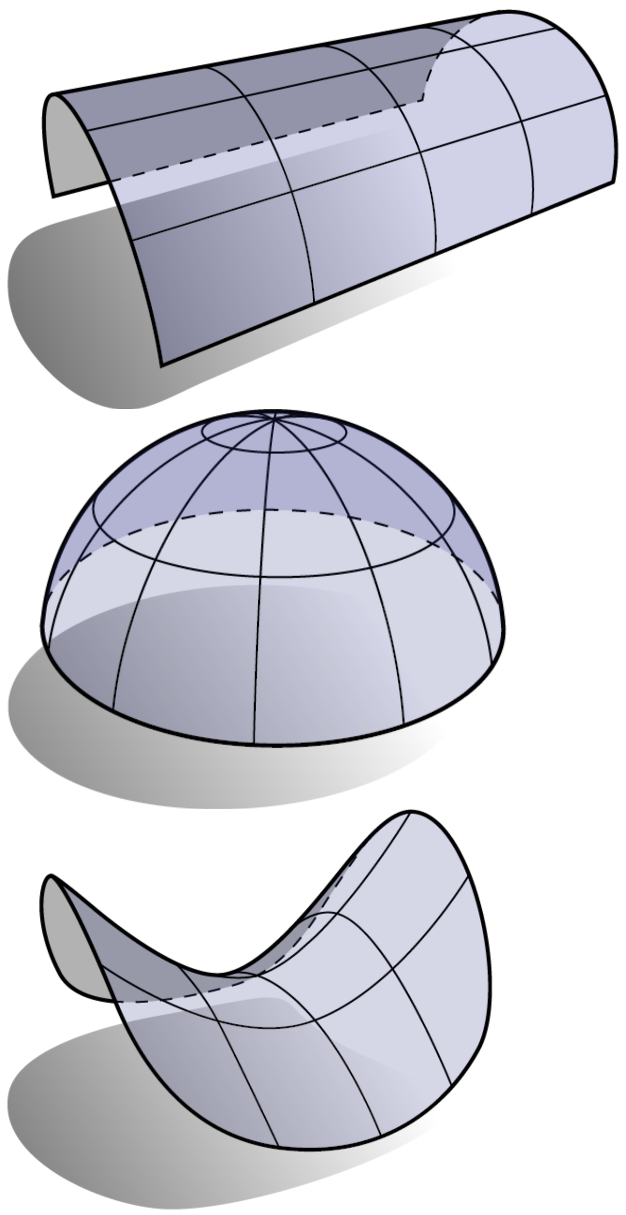
A surface can, locally, be

- Planar: $\kappa_1 = \kappa_2 = 0$

- Cylindrical: $\kappa_1\,\kappa_2 = 0, \kappa_1 \neq 0$ or $\kappa_2 \neq 0$

- Elliptic: $\kappa_1\,\kappa_2 > 0$

- Hyperbolic: $\kappa_1\,\kappa_2 < 0$



Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus. ACM SIGGRAPH 2013

# Mean and Gauss curvatures

- Gauss curvature: $K_S = \kappa_1\,\kappa_2 = \det(\mathrm{W}_S) = \dfrac{\det(\mathrm{II}_S)}{\det(\mathrm{I}_S)}$

- Mean curvature: $H_S = \dfrac{1}{2}(\kappa_1 + \kappa_2) = \dfrac{1}{2}\,\mathrm{tr}(\mathrm{W}_S)$

- $H_S = 0 \Leftrightarrow S$ is a minimal surface
- $K_S = 0 \Leftrightarrow S$ is a developable surface