Affine Transformation

Affine/Linear Transformation

Most used deformations $p' = f(p)$

can be represented by a matrix $\rightarrow$ can be sent as uniform parameter to the shader.

- In standard 3D coordinates $p = (x, y, z)$

$p' = \mathrm{L}\, p + t$

L: linear component (3×3 matrix)

t: translation

- In homogeneous coordinates $p = (x, y, z, 1)$

$p' = \mathrm{A}\, p$

A: 4×4 matrix

$$\mathrm{A} = \left(\begin{array}{c|c} \mathrm{L} & t \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} \mathrm{L}_{xx} & \mathrm{L}_{xy} & \mathrm{L}_{xz} & t_x \\ \mathrm{L}_{yx} & \mathrm{L}_{yy} & \mathrm{L}_{yz} & t_y \\ \mathrm{L}_{zx} & \mathrm{L}_{zy} & \mathrm{L}_{zz} & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Properties:

- $\det(\mathrm{L})$: Change of volume when applying L to a shape
- $\mathrm{L}^T\, \mathrm{L} = 1 \Rightarrow$ Isometry

Affine/Linear Deformations

Translation

$$t(p) = (x + t_x, y + t_y, z + t_z)$$

$$\mathrm{T} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaling

$$s(p) = (s_x\, x, s_y\, y, s_z\, z)$$

$$\mathrm{S} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation

Several possible representations (see later)

Note: Isometry

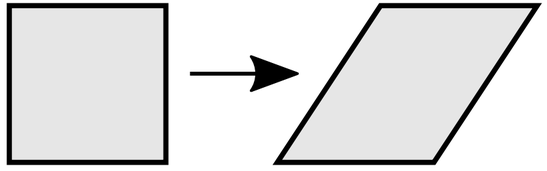
Shearing

$$sh_{xy}(p) = (x + \lambda y, y, z)$$
$$sh_{xz}(p) = (x + \lambda z, y, z)$$
$$sh_{yx}(p) = (x, y + \lambda x, z)$$
$$\ldots$$

$$\mathrm{Sh}_{xy} = \begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathrm{Sh}_{xz} = \begin{pmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathrm{Sh}_{yx} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ldots$$

Usually avoided in Graphics

Note: $\det(\mathrm{Sh}) = 1 \rightarrow$ *constant volume (but no isometry)*



Rotations

3D rotations have 3 dof, No unique representation

Matrix

$$R = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

$$R^T R = \mathrm{I}$$
$$\det(R) = 1$$

(+) Computationaly convenient
(-) Non-explicit dof, redundancies

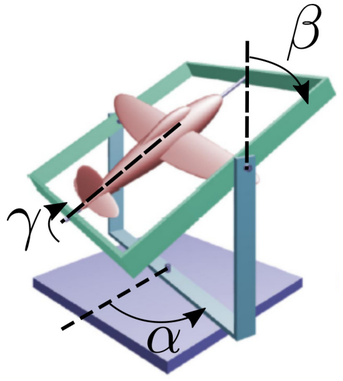
Euler Angles

3 angles: (α, β, γ)

Composition of rotation around basic axes

Not unique (x-y-z, y-z-x, x-y-x', x-z-x', ...)

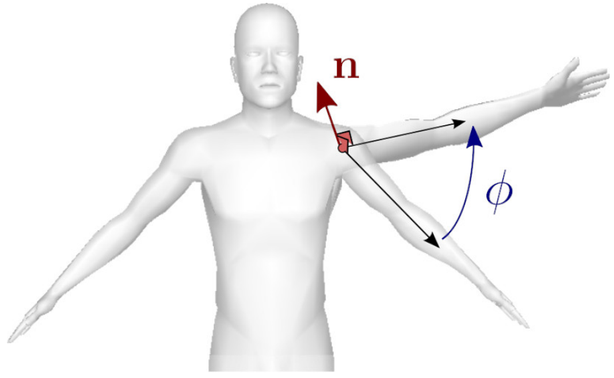
(+) Meaningfull parameters
(-) Gimbal-lock



Axis angle

$(\mathbf{n}, \theta)$

(+) Meaningfull parameters
(-) No direct composition



Quaternion

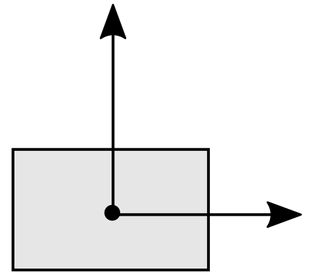
$$q = (x, y, z, w)$$
$$= \left(\mathbf{n} \sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right)\right)$$

(+) Composition and interpolation
(-) Less intuitive components

Cautions with transformations order

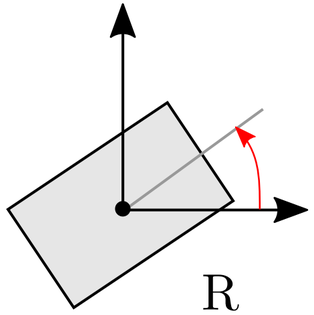
Take care, order of operation does matter !

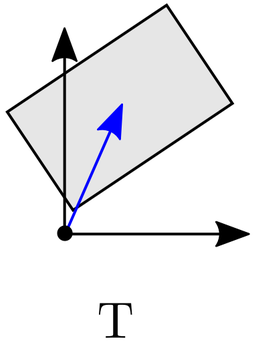
Rotation r, Translation t: $r \circ t \neq t \circ r \Rightarrow \mathrm{M}_1 = \mathrm{T}\,\mathrm{R} \neq \mathrm{R}\,\mathrm{T} = \mathrm{M}_2$



Take care (2): transformation matrices applied to coordinates from right to left.

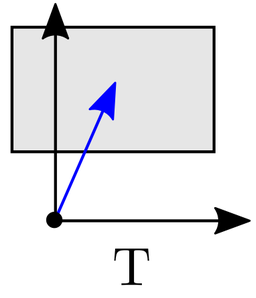
$$\mathrm{M}_1 = \mathrm{TR} = \left(\begin{array}{c|c} 1 & t \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} R & t \\ \hline 0 & 1 \end{array}\right)$$

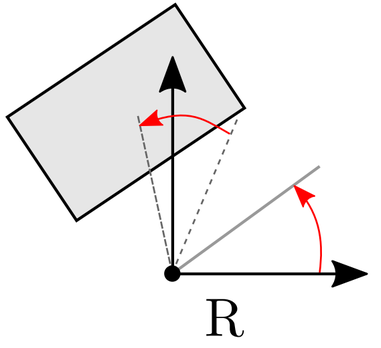




First rotates, then translates

$$\mathrm{M}_2 = \mathrm{RT} = \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} 1 & t \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} R & \mathrm{R}\,t \\ \hline 0 & 1 \end{array}\right)$$





First translates, then rotates

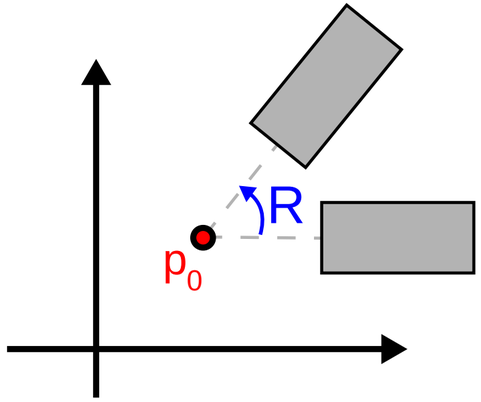
Rotation always happens around the origin.

Beware: Some libraries (old OpenGL, Three.js) apply transformation from "left to right" using transposed matrix multiplication

Affine Transformation Exercises

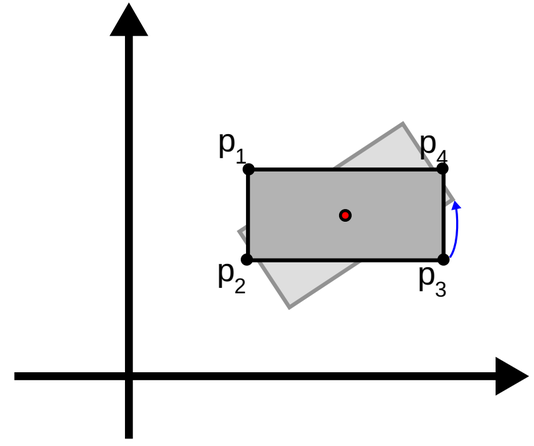
Q. Express the affine transform (as a 4×4 block matrix) corresponding to a rotation R applied around an arbitrary position p_0 in space.

> $M = \ldots$



Q. Consider a shape defined by a triangular mesh with vertex positions $(p_i)_{i \in [1,N]}$.

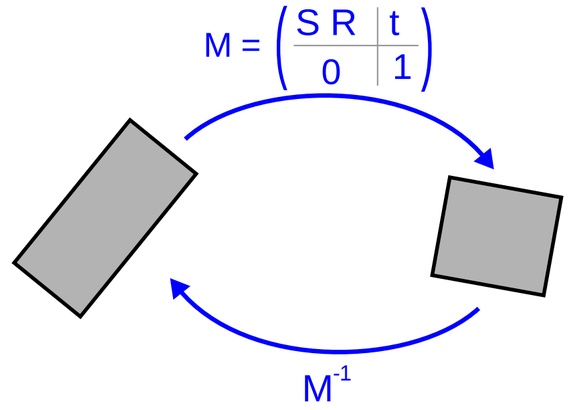
> Express the affine transform allowing to rotate the shape around its barycenter.



Affine Transformation Exercises

Q. Consider the affine transform M parameterized by a scaling s, a rotation R, and a translation t.

> Express the inverse matrix M^{-1} with respect to s, R and t



Interpolating rotation

Do not use componentwise-interpolation on rotation matrix

$\Rightarrow$ interpolate in **quaternion** space

Can use either:

- **SLERP** - Spherical Linear Interpolation

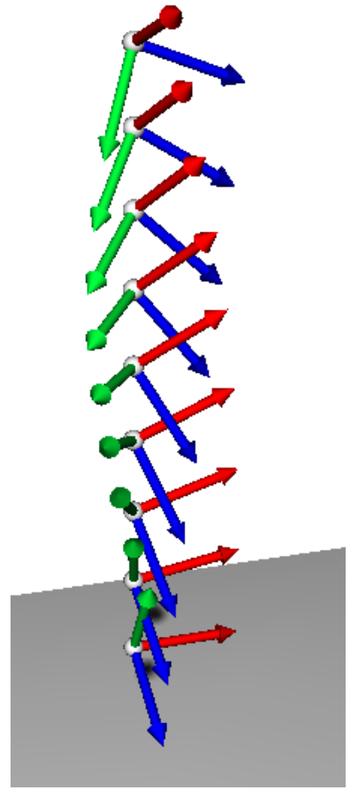
$$q(t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)}\, q_1 + \frac{\sin(t\Omega)}{\sin(\Omega)}\, q_2, \quad \text{with } \cos(\Omega) = q_1 \cdot q_2$$

Between two unit quaternions q_1, q_2

- **LERP** - Linear Interpolation

$$q(t) = \frac{\sum_j \alpha_j\, q_j}{\| \sum_j \alpha_j q_j \|}$$

When blending multiple quaternions q_j *with weights* α_j



Rem.

When interpolating b/w rotations and positions:

Use quaternion with rotation, componentwise-interpolation on position

Handling affine transformation : Polar Decomposition

Key-pose transforms are often given as 4×4 matrices.
How can we interpolate b/w affine transforms?

- Splitting linear M/translation part is easy.
- *Problem*: interpolating the linear part M
  mixes rotation, scaling, shearing

$\Rightarrow$ Split M into: rotation part, and scaling/shearing.

- Interpolate rotation with quaternion (ex. SLERP/LERP)
- Scaling/shearing using componentwise interpolation (ex. linear).

- Polar decomposition: $\mathrm{M} = \mathrm{R}\,\mathrm{D}$
  - R: Rotation matrix
  - D: Positive semi-definite matrix

[K. Shoemake and T. Duff, Matrix Animation and Polar Decomposition. Graphics Interface 92.]

- Polar decomposition is obtained from SVD

$$\mathrm{SVD}(\mathrm{M}) = W \Sigma V^T \text{ with } \boxed{\mathrm{R} = \mathrm{W}\mathrm{V}^T}, \quad \mathrm{D} = \mathrm{V}\Sigma\mathrm{V}^T$$

- Or numerically, R can be computed using the following iterative scheme

$$\boxed{R_0 = \mathrm{M}\,,\ R_{i+1} = 0.5\,(R_i + (R_i^{-1})^T)}$$

Handling affine transformation

Algorithm to interpolate between two general 4×4 matrices M_1, M_2

1. Extract translation p_1, p_2 from M_1, M_2.
2. Compute R_1, R_2 ($\mathbf{3} \times \mathbf{3}$ rotation matrices), and D_1, D_2 ($\mathbf{3} \times \mathbf{3}$ scaling/shearing matrices) from M_1, M_2
3. Interpolate linearly position and scaling/shearing $p(t) = (1-t)\, p_1 + t\, p_2$, $D(t) = (1-t)\, D_1 + t\, D_2$
4. Compute quaternions q_1, q_2 from R_1, R_2

   Note $M \to q$ with $q = \left(\frac{M_{zy} - M_{yz}}{2r}, \frac{M_{xz} - M_{zx}}{2r}, \frac{M_{yx} - M_{xy}}{2r}, \frac{r}{2} \right)$, $r = \sqrt{1 + M_{xx} + M_{yy} + M_{zz}}$
5. Compute $q(t) = \text{SLERP}(q_1, q_2, t)$
6. Convert back to matrix $q(t) \to R(t)$
7. Compute final matrix $M(t) = R(t)\, D(t)$ with translation $p(t)$.

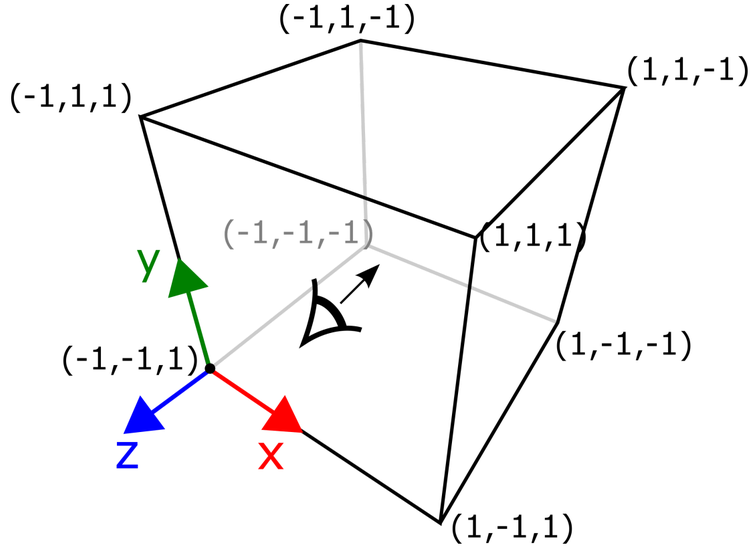
Camera and OpenGL

In OpenGL: no real notion of *"camera"*

Unique viewpoint - cube $(x, y, z) \in [-1, 1]^3$ *- Normalized Device Coordinate*

Left-right/x, Bottom-up/y, Front-back/z (toward negative z)

No perspective (displays (x,y) for the smallest z)



Any *camera effect* (position, orientation, perspective) must be coded "manually" in the shader.

All these effects can be represented as matrices.

Three common matrices

Model matrix

Affine transform - specific to a shape.

Local object coordinates to world space.

View matrix

Rigid transform - position/orientation of the camera in the scene.

World space to camera/view/eye space

Projection matrix

Perspective matrix - map frustum to normalized cube. (/or ortho projection)

Camera space to Normalized Device Coordinate (clip/screen space)

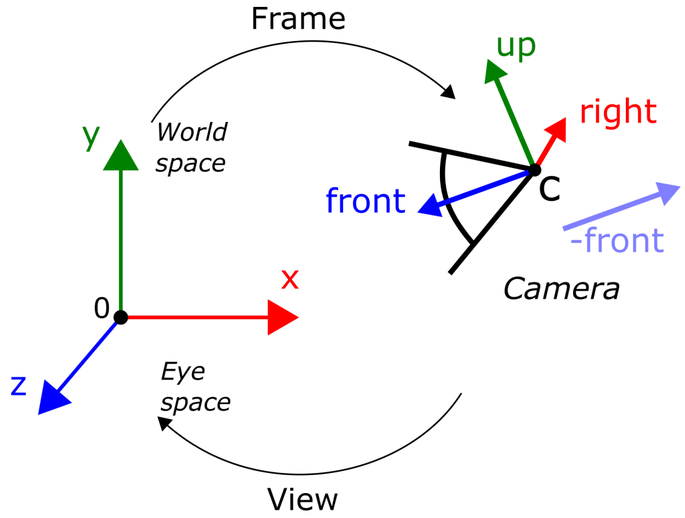
$$p = Projection \times \underbrace{View \times Model}_{ModelView} \times p_0$$

View representation as matrix

Camera parameterized by:

A center c; right, up, front unit vectors

expressed in global coordinates



Frame matrix:

Transform unit coordinates into camera vectors

$Frame \times (1,0,0,0) = right$
$Frame \times (0,1,0,0) = up$
$Frame \times (0,0,1,0) = -front$
$Frame \times (0,0,0,1) = c$

Matrix of column-oriented vectors of camera directions

$$\text{Frame}=\begin{pmatrix} | & | & | & | \\ right & up & -front & c \\ | & | & | & | \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} O & c \\ 0 & 1 \end{pmatrix}$$

View matrix: inverse of the Frame matrix

Transforms the camera vectors into the unit coordinates

$View \times right = (1,0,0,0)$
$View \times up = (0,1,0,0)$
$View \times -front = (0,0,1,0)$
$View \times c = (0,0,0,1)$

Orientation O can be read on the rows

$$\text{View}=\begin{pmatrix} (\cdots & right & \cdots) & | \\ (\cdots & up & \cdots) & -O^T c \\ (\cdots & -front & \cdots) & | \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Camera in shader - Question

Q. In a vertex shader the View matrix is passed as uniform parameter
How can you compute the position of the camera ?

A usefull function to set a point of view is the standard *look_at(eye, center, up)*
- eye: position of the viewer
- center: position that the viewer is looking at
- up: a direction representing the "vertical direction" of the viewer.

Q. How to compute the *View* matrix from the *look_at* parameters ?

Interacting with camera

Camera orientation has 3 degrees of freedom (+ 3 dof in position)

How to handle a camera orientation using a mouse (2-dof) ?

Assuming a freely oriented camera

Common approaches

"Spherical camera"/coordinates (θ, ϕ)

(+) Good to easily "turn around" oriented structure

(-) Lacks a dof: No "twist": position on the unit sphere force the *angle*
Not adapted to turn around non-oriented/badly-oriented objects.

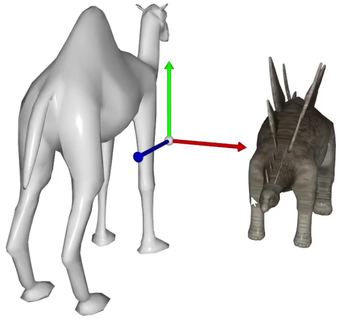
Spherical coordinates + twist (ex. roll, pitch, yawn)

(+) Precise and complete control

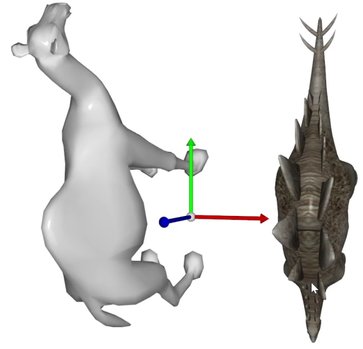
(-) Require an additional key/controler to the mouse to control the 3 dof

Example of "Spherical camera"

Correct orientation



Bad orientation of the shape
spherical camera is "stuck" in this orientation



Interacting with camera - ArcBall/Trackball metaphore

Other approach: **ArcBall/Trackball behavior**

3 dof: left/right, top/down, but also twist

Idea: Use the 2D cursor position as "3D point on the trackball"

Motion between 2 cursor positions = 3D Rotation applied to the ball

[K. Shoemake. A User Interface for Specifying Three-Dimensional Orientation Using a Mouse. Graphics Interface, 1992.]

Algorithm:

Inputs $p_1 = (p_{1x}, p_{1y}), p_2 = (p_{2x}, p_{2y})$ in screen coord.

$q_{1/2}$ = ArcBallProjection$(p_{1/2})$

R= rotation between vectors (q_1, q_2)

ArcBallProjection(p)

d =norm(p)

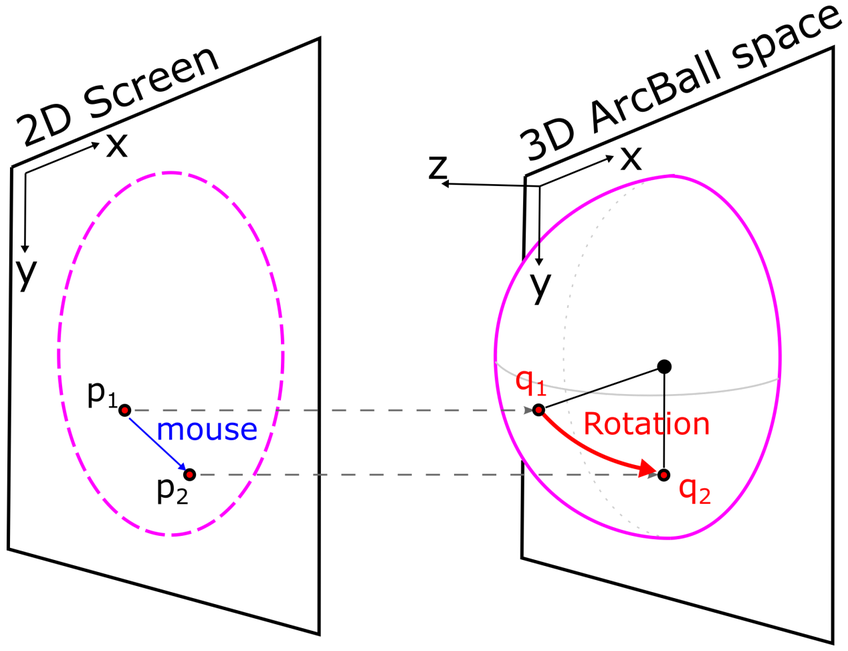
if(d<$1/\sqrt{2}$)

$q = (p, \sqrt{1-d^2})$

else // hyperbola

$q = (p, 1/(2d))$

return q



Note: the hyperbola smoothly extends the sphere in the entire screen.

Interacting with camera - ArcBall/Trackball metaphore

Natural way to rotate around a shape

(+) No privilegiated axis

No reference configuration, works in incrementing rotations

Behavior is orientation invariant

(-) Less precise than individual dof control

Drift of twist when moving back and forth