Shape deformation (III)

Surface-based deformation

Laplacian deformation
ARAP - As Rigid As Possible

Spatial / Surface based deformation

Consider a shape defined as $(p_i)_{i \in [0, N-1]}$ positions

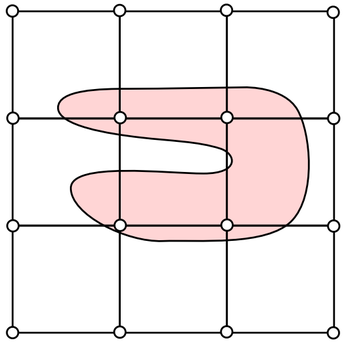
Spatial/volume deformation

Define spatial deformation $\forall p \in \mathbb{R}^3, \quad f : p \to f(p)$

Apply f to every positions

$\forall i \in [0, N-1], \; q_i = f(p_i)$

(+) Independant from shape representation
Point sets, Surface, Volume.
(-) Indirect control
Spatial deformation $\to$ shape deformation
(-) Deformation fully defined by spatial position



Surface-based

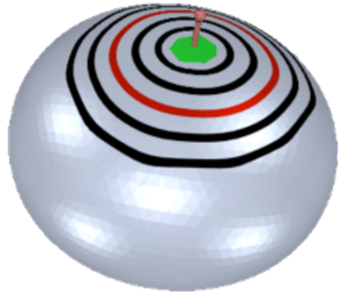
Compute deformation on surface only

Deformation is defined at p_i

But not at arbitrary position in space

ex. $q_i = p_i + n_i, \quad n_i$:surface normal at position p_i

(+) Can integrate/preserve surface properties (neighborood, curvature, etc.)
(-) Sensible to surface representation & connectivity.



Laplacian/Differential Coordinates Editing

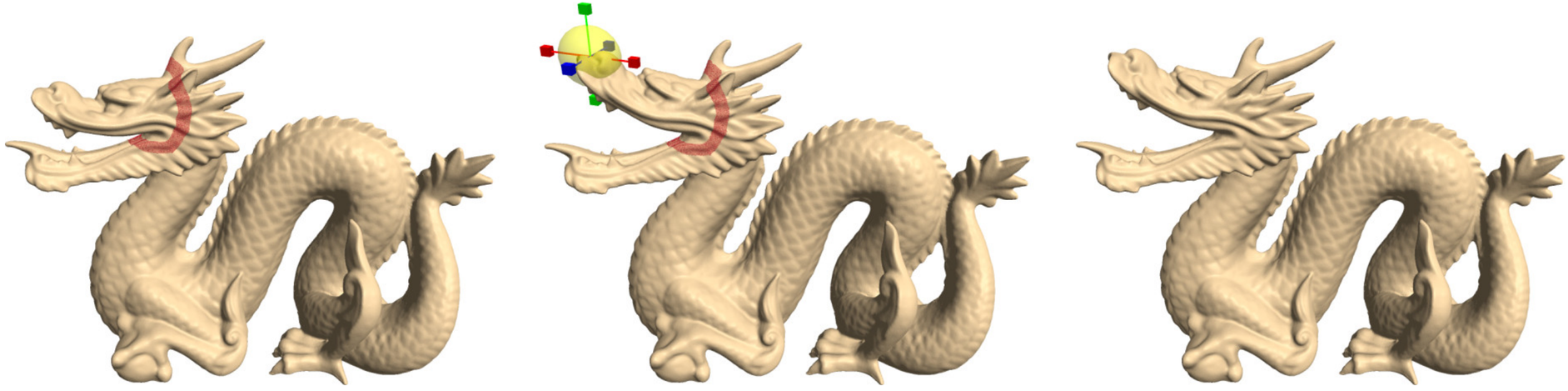
Initialization: Encode vertex position as *differential coordinates* w/r neighbors

Ex. Laplacian coordinate value δp

During deformation

- Try to match constrained position for subset of vertices
- While preserving the differential coordinates

Local neighborhoods appearance



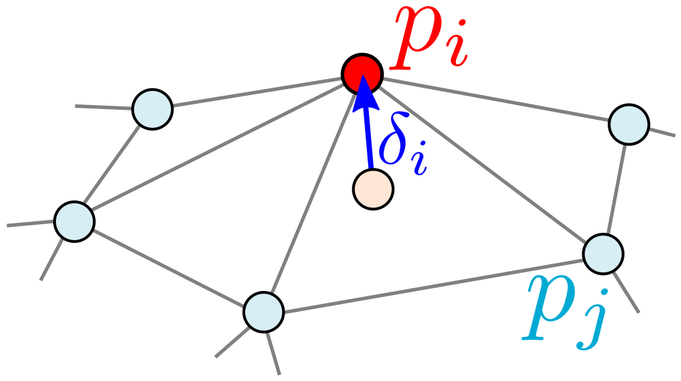
Laplacian of the coordinates

- For a scalar/vector function f defined in $\mathbb{R}^2$ with parameter (u, v): $\triangle f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = div(grad(f))$
- Laplacian on a manifold: Called Laplace-Beltrami operator.
- Differential coordinates δ: Laplace-Beltrami operator applied on coordinates functions $f = (x, y, z)$ itself.

Common approximation in the discrete case

$$\delta_i = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} (p_i - p_j) = p_i - \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} p_j$$

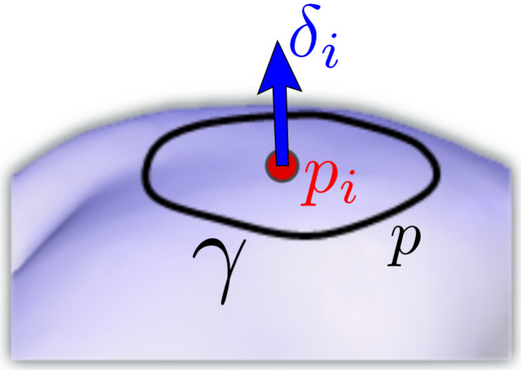
$\mathcal{N}_i$: one ring neighborhood



Note: $\delta_i \simeq \lim_{|\gamma| \to 0} \frac{1}{|\gamma|} \int_{p \in \gamma} (p_i - p)\, \mathrm{d}l(p) = \mathrm{H}(p_i)\, n_i$

γ: small contour around p_i

$\mathrm{H}(p_i)$: Mean curvature at p_i, n_i normal at p_i



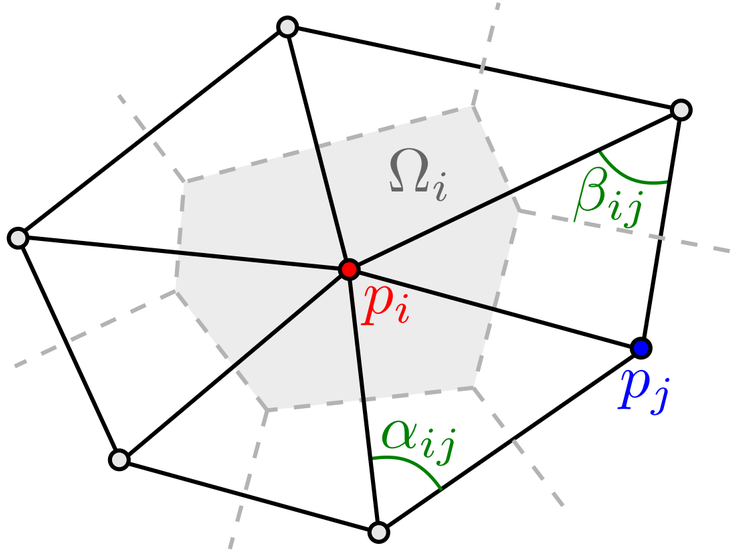
$\Rightarrow$ Encode an approximation of the mean curvature

Laplacian of the coordinates: Approximation

Simplest approximation of the laplacian (Graph-Laplacian): $\delta_i = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} (p_i - p_j)$

(-) *Depends on the connectivity, assume homogeneous and regular geometrical sampling*

Other possible approximations: $\delta_i = \sum_{j \in \mathcal{N}_i} \omega_{ij} \, (p_i - p_j)$



Ex. Cotangent weights ("Mesh Laplacian")

$$\delta_i = \frac{1}{|\Omega_i|} \sum_{j \in \mathcal{N}_i} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) \, (p_i - p_j)$$

Better geometrical properties: independant of the connectivity

(-) Can become negative

Or using mean-value coordinates

No perfect approximation (satisfying all properties of the continuous Laplacian operator)

[Discrete Laplace operators: No free lunch. M. Wardetzky et al. SGP 2007]

Laplacian deformation formulation

Given initial position $(p_i)_{i \in [0, N-1]}$ + set of N_c positional constraints c_i, $i \in C$.

Find q_i such that

$\delta(q_i) = \delta(p_i)$: Same differential coordinates

$\forall i \in C, q_i = c_i$: Positional constraints

Formulate using quadratic energy w/r to unknown q

$$E = \sum_{i=0}^{N-1} \|\delta(q_i) - \delta(p_i)\|^2 + \omega \sum_{i \in C} \|q_i - c_i\|^2$$, w: weight associated to positional constraints.

$$E = \sum_{i=0}^{N-1} \left\| q_i - \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} q_j - \delta(p_i) \right\|^2 + \omega \sum_{i \in C} \|q_i - c_i\|^2$$

Laplacian deformation least square formulation

$$E = \sum_{i=0}^{N-1} \left\| q_i - \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} q_j - \delta(p_i) \right\|^2 + \omega \sum_{i \in C} \|q_i - c_i\|^2$$

Finding q minimizing E is similar to solve the least square problem

$$q^\star = arg\min_q \ \|\mathrm{M}\, q - b\|^2$$

$q = (q_0, \cdots, q_{N-1})^T$: N unknown

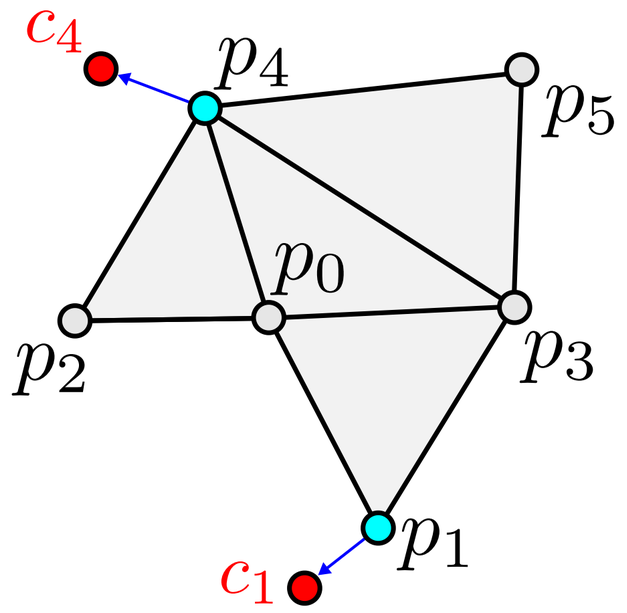
$\mathrm{M} = \begin{pmatrix} \mathrm{D} \\ \mathrm{C} \end{pmatrix}$ with D ($N \times N$) ref. Laplacian weights; C ($N_c \times N$) ref. constraints.

- $\forall i \in [0, N-1]$, $\mathrm{D}_{ii} = 1$, $\mathrm{D}_{ij} = -1/|\mathcal{N}_i|$ if $j \in \mathcal{N}_i$
- $\forall k \in [0, N_c - 1]$, $\mathrm{C}_{ki} = \omega$, with i refering to the vertex index of the kth constraint.

$b = (\delta(p_0), \cdots, \delta(p_{N-1}),\ \omega\, c_0, \cdots, \omega\, c_{N_c-1})^T$

Example of matrix

$$\underbrace{\left(\begin{array}{cccccc} X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ \hline X & X & X & X & X & X \\ X & X & X & X & X & X \end{array}\right)}_{\mathrm{M}} \underbrace{\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}}_{q} = \underbrace{\left(\begin{array}{c} \delta_0 \\ \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \hline \omega\, c_1 \\ \omega\, c_4 \end{array}\right)}_{b}$$



Q. Fill the values of the matrix for this case

Note: Only 2 soft positional constraints on q_1 and q_4

Laplacian deformation least square solution

Need to minimize $||\mathrm{M}\,q - b||^2$

Similar to solve the linear system $(\mathrm{M}^T\,\mathrm{M})\,q = \mathrm{M}^T\,b$ (*Normal equations*)

$\mathrm{M}^T\mathrm{M}$ is sparse (diagonally dominant)

Can use standard sparse solver (CG, Jacobi, Gauss Seidel, etc.)

Reminder: Never try to compute the inverse matrix ! Only solve the linear system.

Special solvers (QR factorization) doesn't even need to compute the normal equations

System can be solved independently on x, y, z coordinates. Same matrix M

$$\mathrm{M}\,q_x = b_x, \quad \mathrm{M}\,q_y = b_y, \quad \mathrm{M}\,q_z = b_z$$

Changing constraints position c_i only impact rhs b

$\rightarrow$ M can be factorized once, and reused during interactive deformation

Changing constrained vertex impact M

$\rightarrow$ need to be re-factorized

Solving a dense least square problem

```cpp
#include <iostream>
#include <Eigen/Dense>

int main()
{
    // Build matrix
    Eigen::MatrixXf M;
    // M.resize(N,N)
    // Fill the matrix ... M(i,j) = value
    M = Eigen::MatrixXf::Random(3, 3); // example

    //QR Decomposition (different types are proposed in Eigen)
    auto solver = M.colPivHouseholderQr();

    // Build RHS
    Eigen::VectorXf b;
    // b.resize(N); b(i) = value ...
    b = Eigen::VectorXf::Random(3);

    // Solve linear system
    Eigen::VectorXf x = solver.solve(b);

    std::cout << x << std::endl;
    std::cout <<"Check solution : "<< (M*x-b).norm() << std::endl;

    return 0;
}
```

Solving a sparse least square problem

```cpp
#include <iostream>
#include <Eigen/Sparse>

int main()
{
    // Build matrix
    Eigen::SparseMatrix<float> M;

    // Resize and reserve space to store coefficients
    //   ex. N rows, 10 coeffs per rows
    int N = 50;
    M.resize(N,N);
    M.reserve(Eigen::VectorXi::Constant(N,10));
    // Fill the matrix ... M.coeffRef(i,j) = value
    for(int k=0; k<N; ++k) {
        M.coeffRef(k,k)=1.0f; M.coeffRef(k,(k+1)%N)=-0.3f;
    }

    // Compress the matrix representation
    M.makeCompressed();

    // Factorization for a solver (here Conj. Gradient)
    Eigen::LeastSquaresConjugateGradient< Eigen::SparseMatrix<float> > solver;
    solver.compute(M);
    solver.setTolerance(1e-6f);

    // Build RHS
    Eigen::VectorXf b;
    b = Eigen::VectorXf::Random(N);

    // Solve linear system
    Eigen::VectorXf x = solver.solve(b);
    // or solver.solveWithGuess(b, x0)
    //   faster with good guess
    std::cout <<"Check solution : "
       << (M*x-b).norm() << std::endl;
    std::cout <<"N iterations : "
       << solver.iterations() << std::endl;
    return 0;
}
```

Laplacian deformation: Results

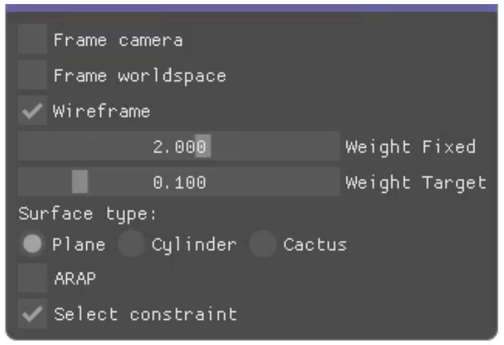
Rem. Discrete solution of the Poisson equation with Dirichlet boundaries

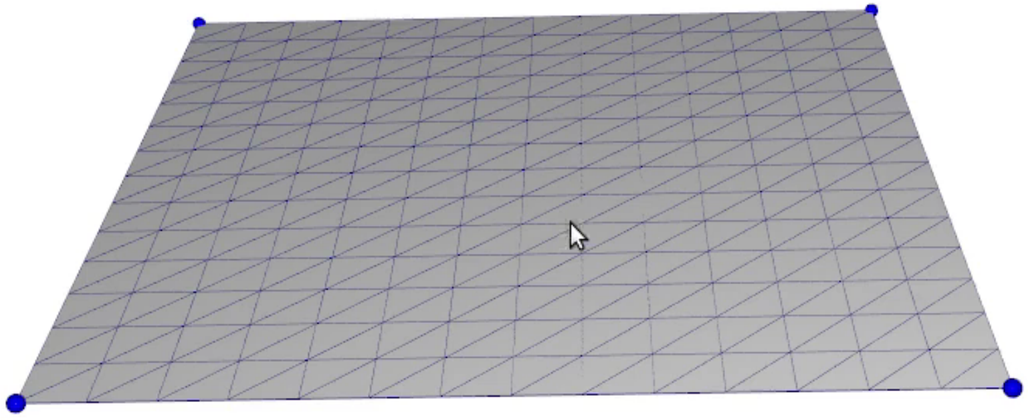
$$\triangle f = \varphi$$
$$f = f^0 \text{ on } \partial\Omega$$

In the case of a plane: Laplacian equation $\triangle f = 0$

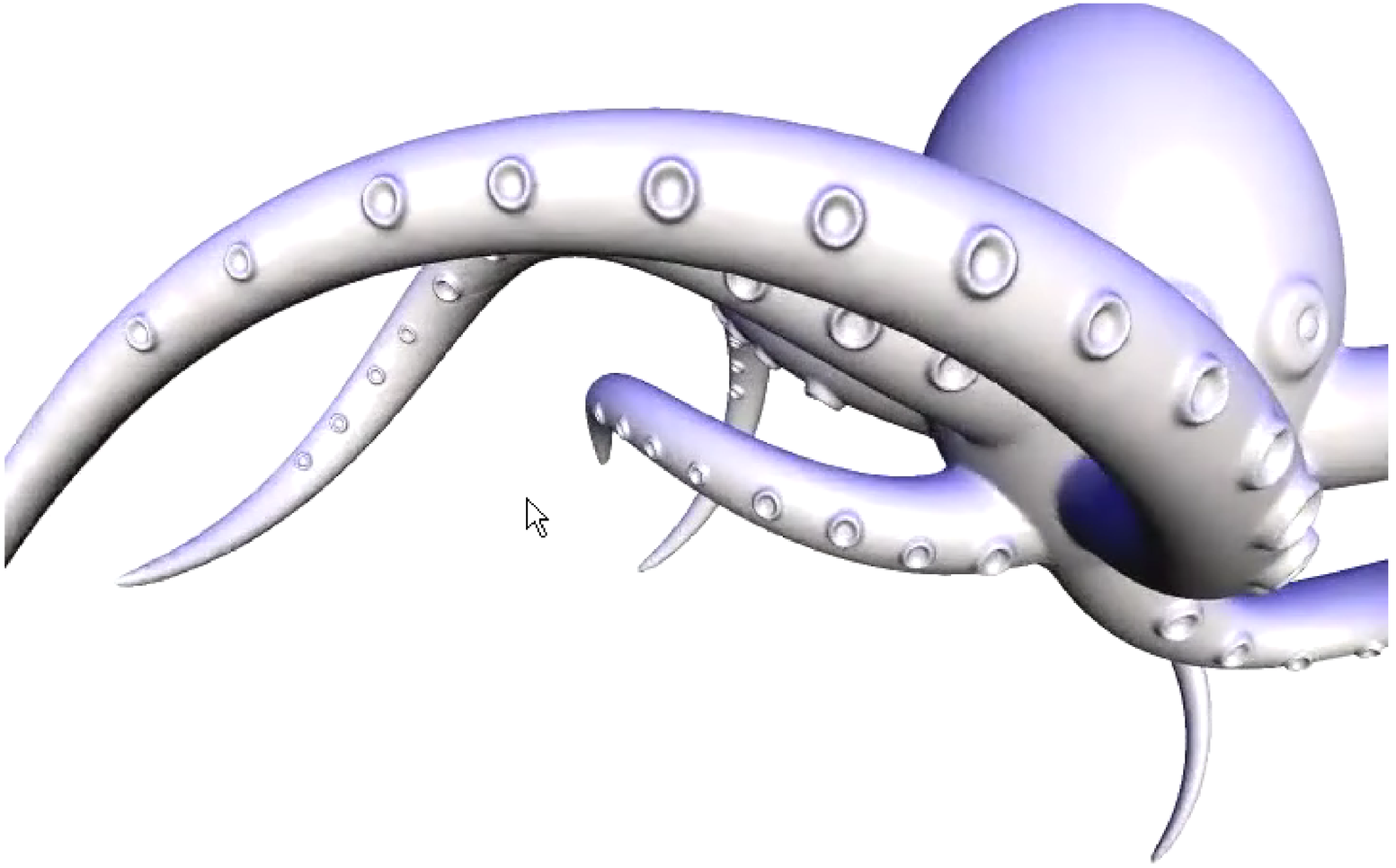
Solution are membranes

Surface of minimal mean curvature





Laplacian deformation: Results



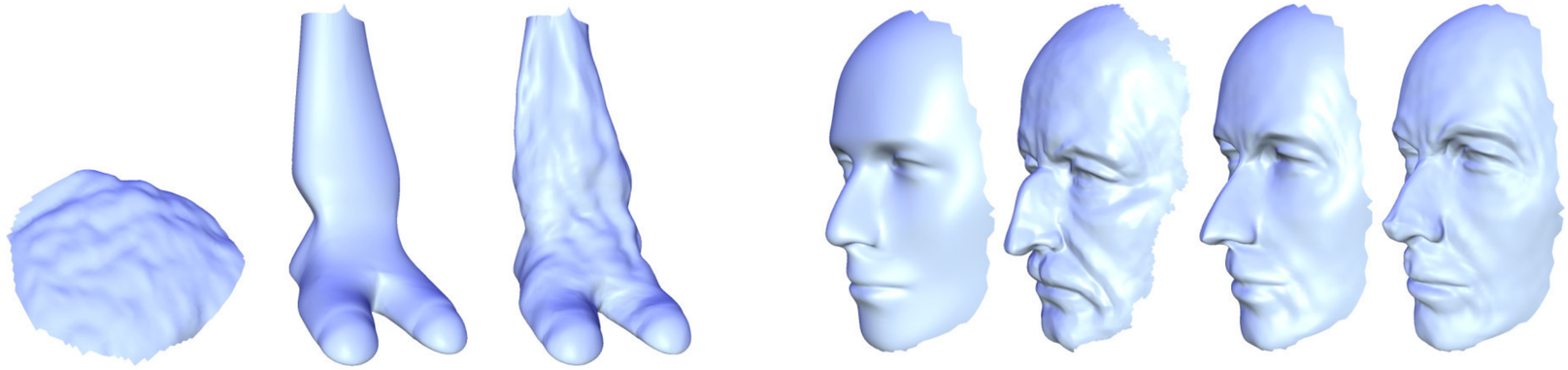
[Laplacian Surface Editing. O. Sorkine et al. SGP 2004]

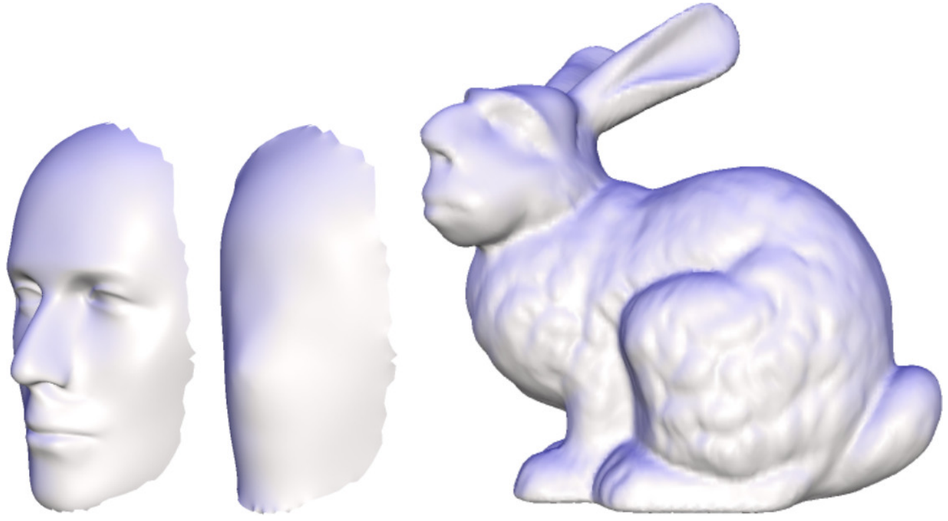
Extension: Detail transfert

Use laplacian from another surface to transfert details

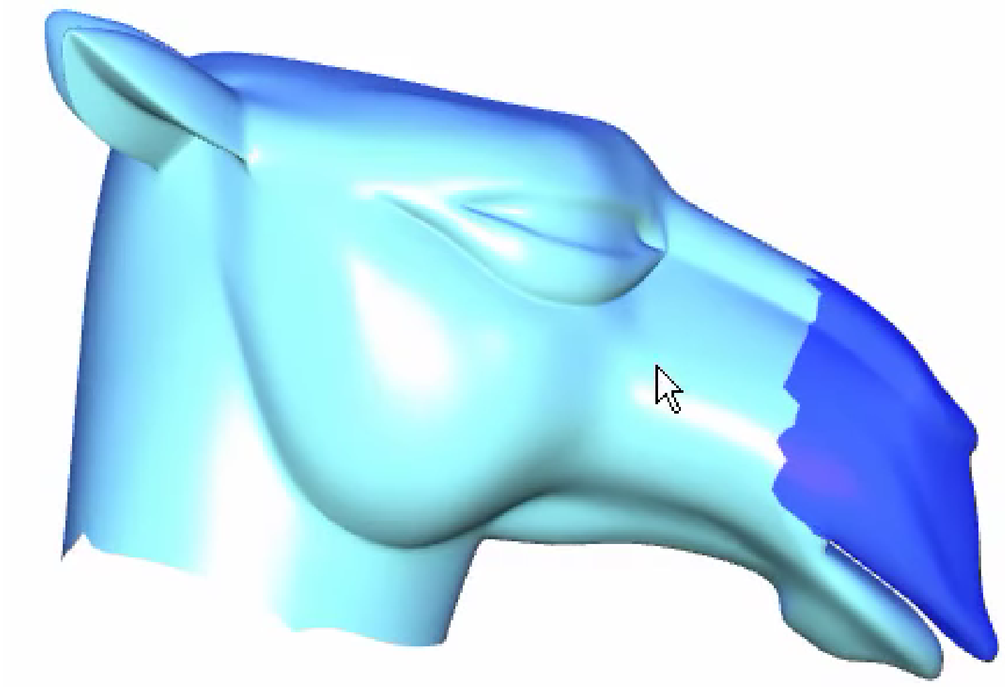
Requires a per-vertex correspondance

User defined anchors





Laplacian editing: Sketching silhouette deformation

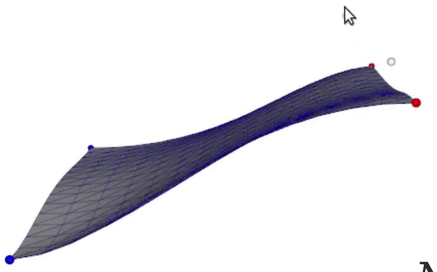


[A Sketch-Based Interface for Detail-Preserving Mesh Editing. A. Nealen et al. SIGGRAPH 2005]

As Rigid As Possible (ARAP)

The Laplacian formulation tries to preserve the surface orientation.

While we could expect surface rotation



No ARAP

Idea: Include rotation transformation R in the formulation

$$E = \sum_{i=0}^{N-1} \|\delta(q_i) - R(q_i)\,\delta(p_i)\|^2 + \omega \sum_{i \in C} \|q_i - c_i\|^2$$

$R(q_i)$: best rotation at point q_i

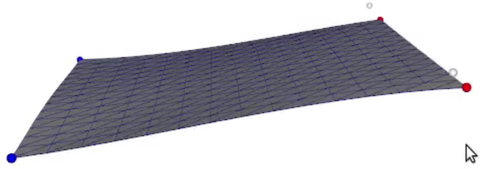
Finding the best rotation at a given point

1- Look at surrounding edges at the point q_i / p_i

$e_j = q_j - q_i, e_j^0 = p_j - p_i$

2- Compute covariance matrix $\sigma = \sum_j e_j \, (e_j^0)^T$

3- Compute rotation using polar decomposition: $R = polar(\sigma)$



ARAP

As Rigid As Possible (ARAP) Solution

Minimize the following term

$$E = \sum_{i=0}^{N-1} \|\delta(q_i) - R(q_i)\,\delta(p_i)\|^2 + \omega \sum_{i \in C} \|q_i - c_i\|^2$$

Problem: $R(q_i)$ is a non linear function of q_i

E is non quadratic, cannot use least square.

Idea: Interleave and iterate on these two steps

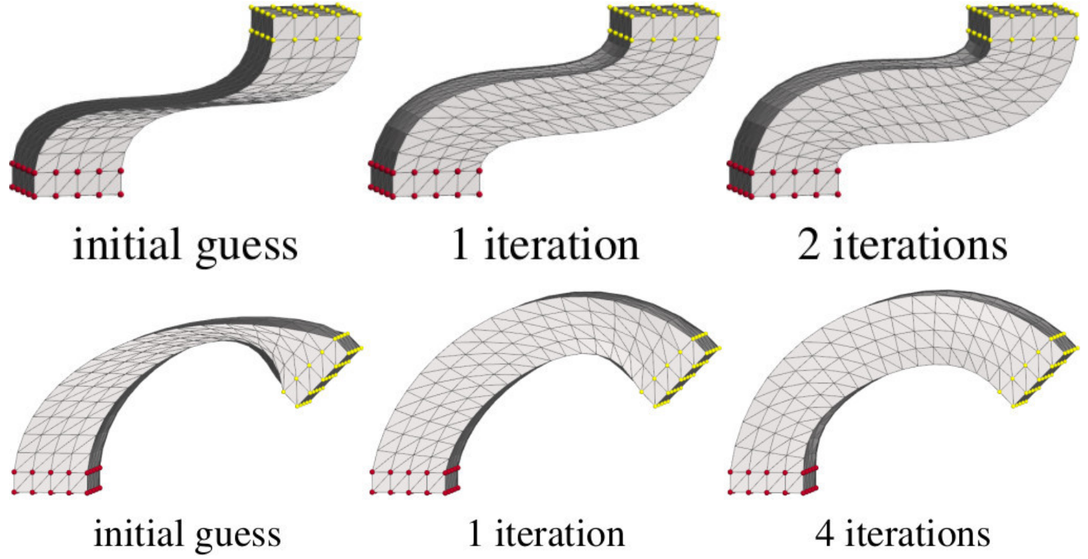
Find optimal rotation $R(q_i)$ with fixed q_i

Find q_i optimizing E with fixed $R(q_i)$

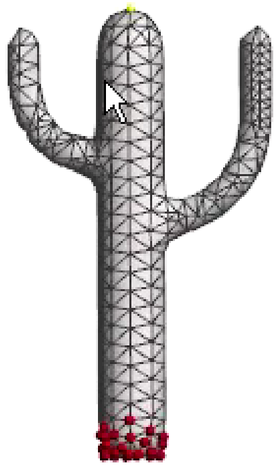
Gain: Surface rotates during deformation

Limit: Slightly more costly

Iterations + polar decomposition



As Rigid As Possible: Results



[link:assets/arap.pdf[As-Rigid-As-Possible Surface Modeling. O. Sorkine, M. Alexa. SGP 2007]]

Other approaches

Gradient representation

[Mesh editing with Poisson-based gradient field manipulation. Y.Yu et al. SIGGRAPH 2004]

Bi-Laplacien weights

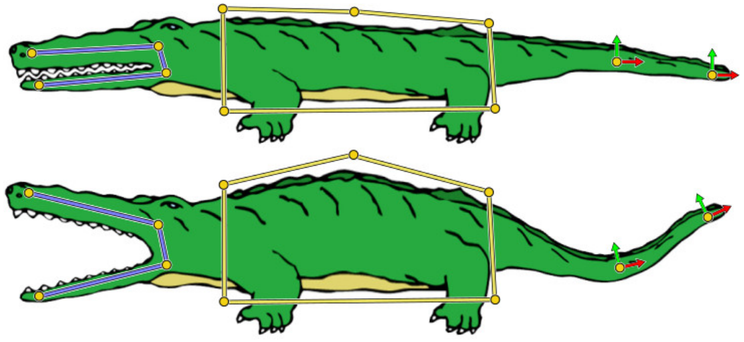
[Bounded Biharmonic Weights for Real-Time Deform. A. Jacobson et al. SIGGRAPH 2011]

Local frame representation

[Linear Rotation-invariant Coordinates for Meshes. Y. Lipman et al. SIGGRAPH 2005]

Deformation propagation

[Harmonic Guidance for Surface Deformation. R. Zayer et al. EG 2005]



State of the art

[On Linear Variational Surface Deformation Methods. M. Botsch and O. Sorkine. TVCG 2008]