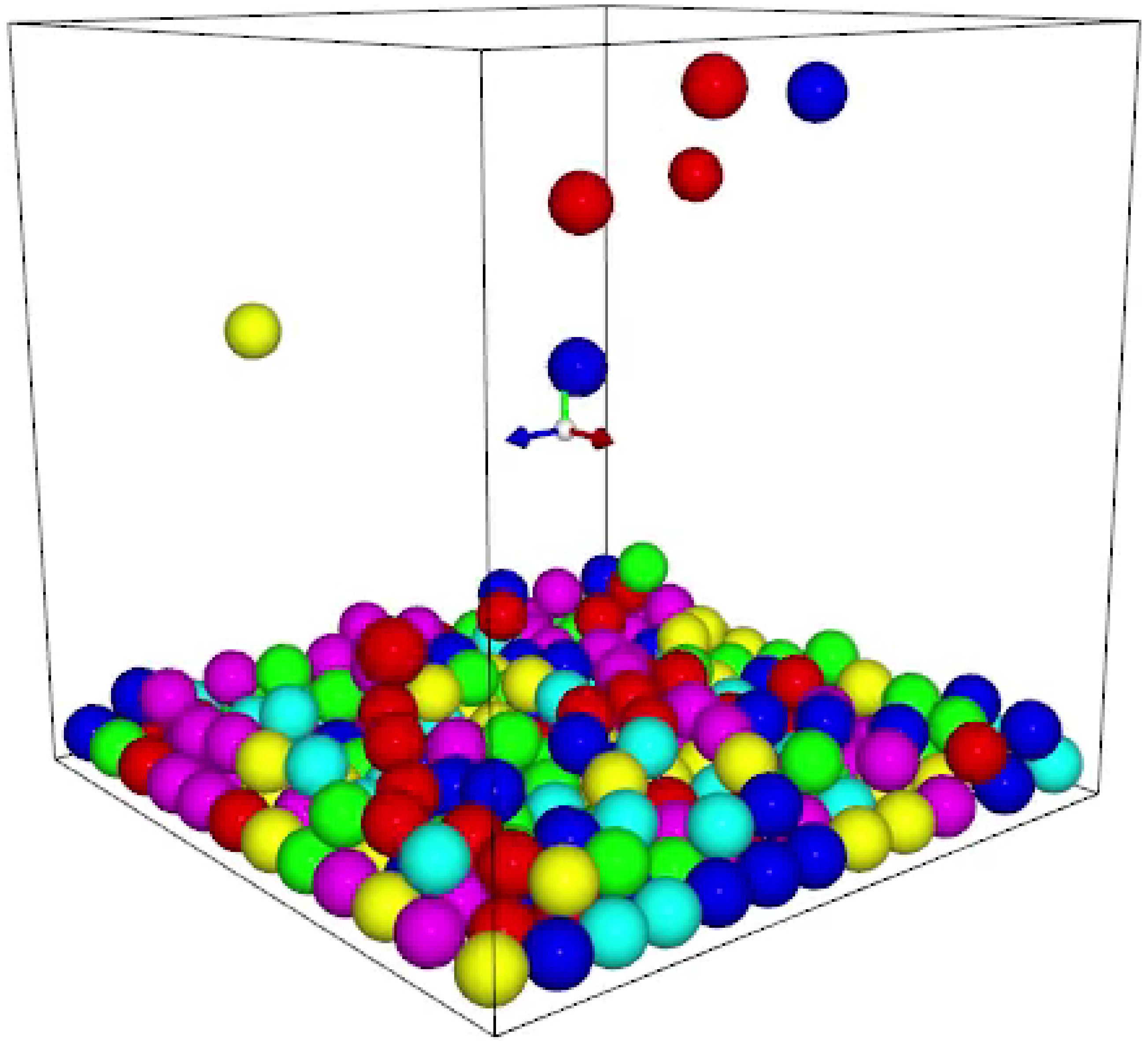
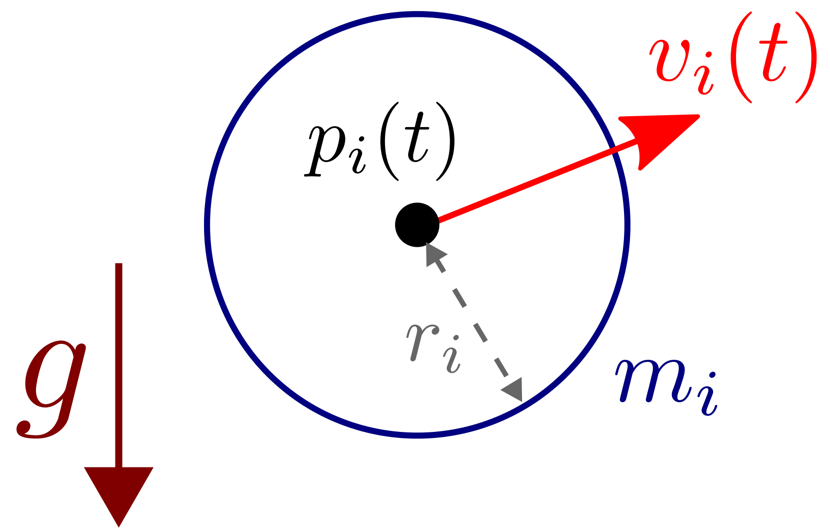
Rigid spheres



System modeling

Particles modeling the center of hard spheres.

- Spheres can collide with surrounding obstacles
- Spheres can collide with each others



- *System*: N particles with position p_i, velocity v_i, mass m_i, modeling a sphere of radius r_i.
  - Initial conditions $p_i(0) = p_i^0, v_i(0) = v_i^0$

- *Forces*: Single gravity forces $F_i = m_i\, g$. Collisions handled by *impulses*.

- *Temporal evolution*: Fundamental principle of dynamics $v_i(t) = p_i'(t), v_i'(t) = g$

- *Numerical solution*

$$\begin{cases} v^{k+1} = v^k + h\, g \\ p^{k+1} = p^k + h\, v^{k+1} \end{cases}$$

Collision with a plane

Plane $\mathcal{P}$: parameterized using a point a and its normal n.

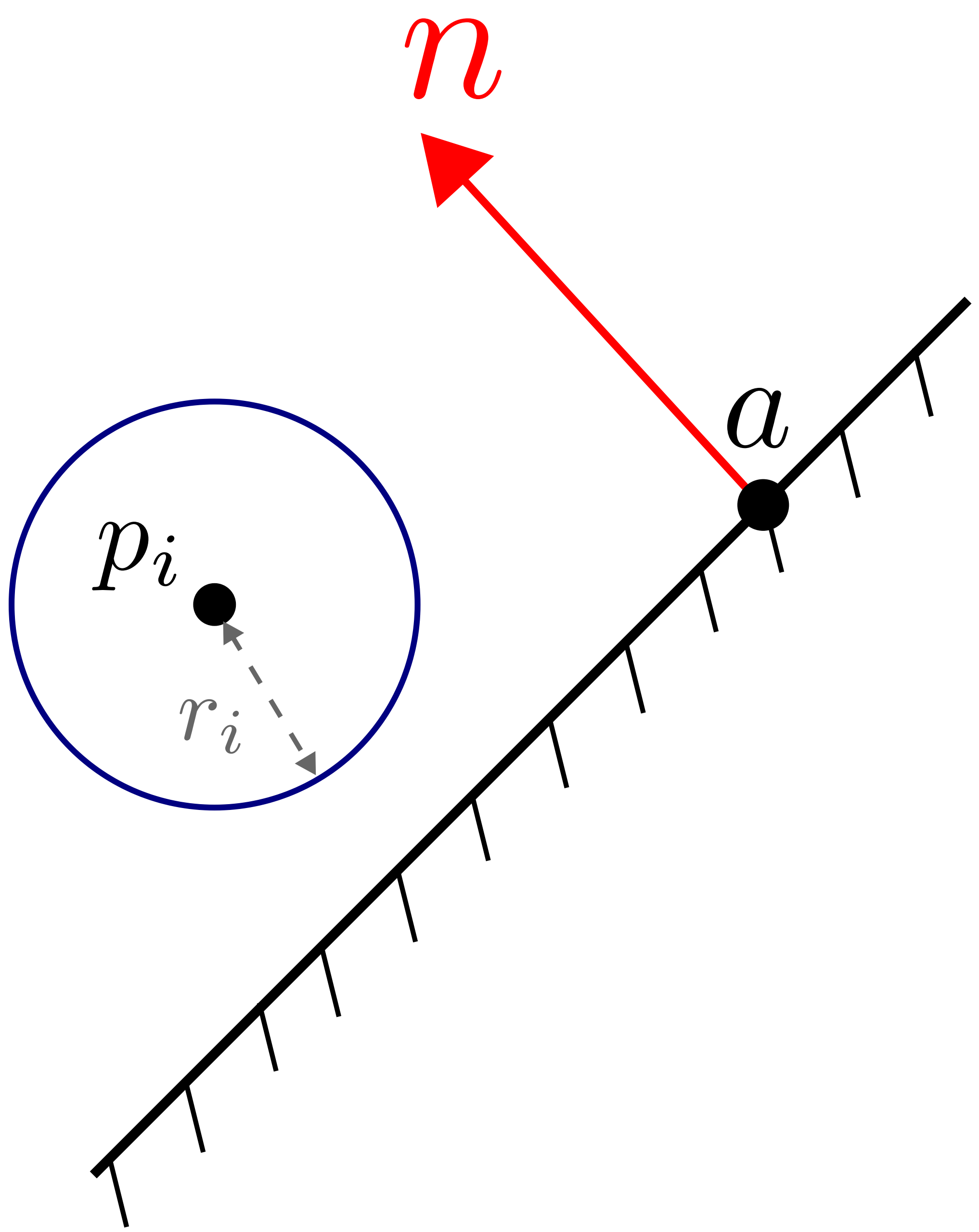
$$\{p \in \mathbb{R}^3 \in \mathcal{P} \Rightarrow (p-a)\cdot n = 0\}$$

- Sphere above plane : $(p_i - a)\cdot n > r_i$
- Sphere in collision: $(p_i - a)\cdot n \leq r_i$

- Collision detection algorithm

```
for(int i=0; i<N; ++i)
{
    float detection = dot(p[i]-a, n);
    if (detection <= r[i])
    {
        // ... collision response
    }
}
```

What should we do when a collision is detected



Collision response with plane

Suppose exact contact: $(p_i - a) \cdot n = r_i$

Collision response = **Update velocity**

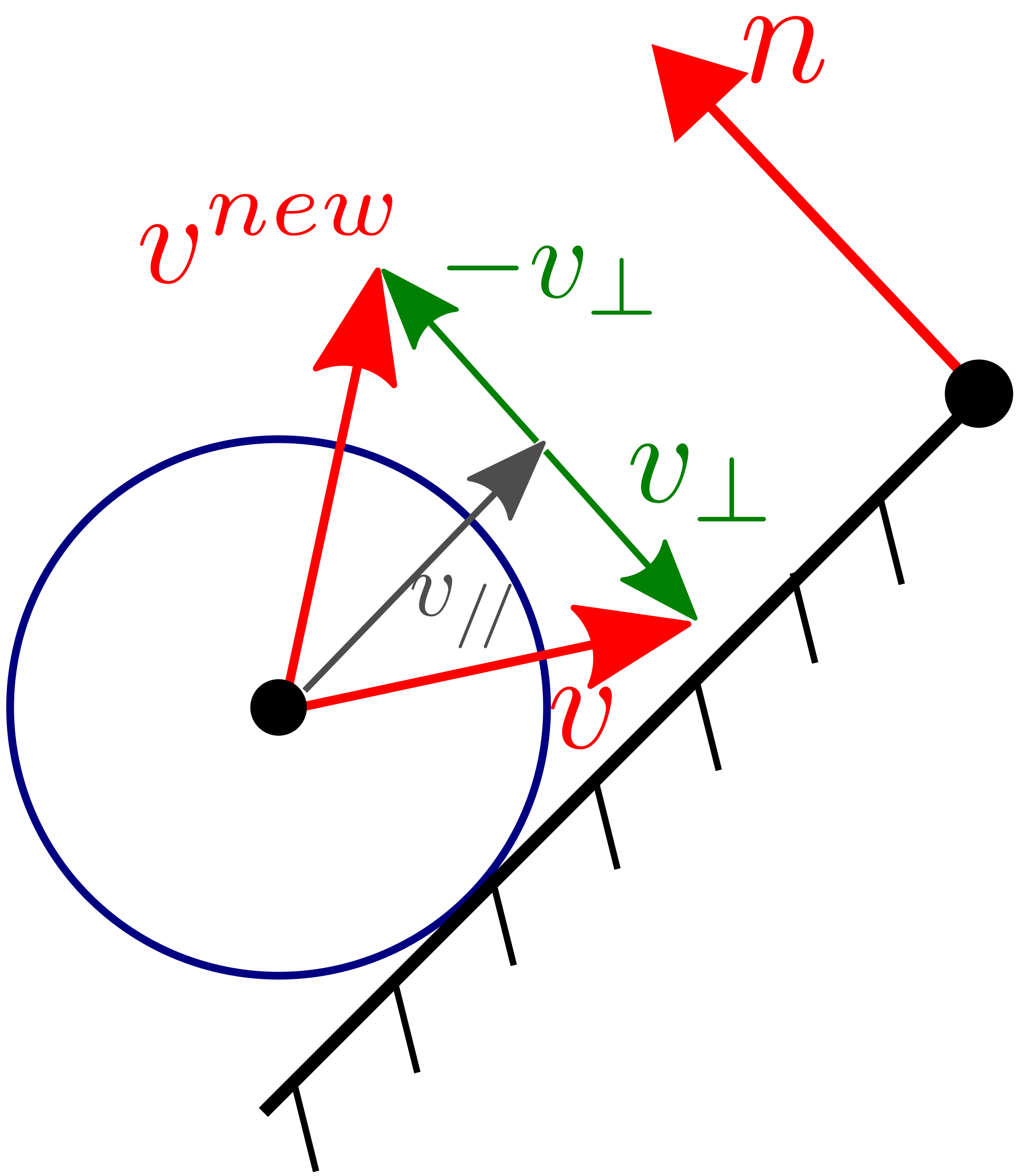
Split $v = v_{//} + v_{\perp}$

- $v_{\perp} = (v \cdot n)\, n$
- $v_{//} = v - (v \cdot n) n$

New velocity

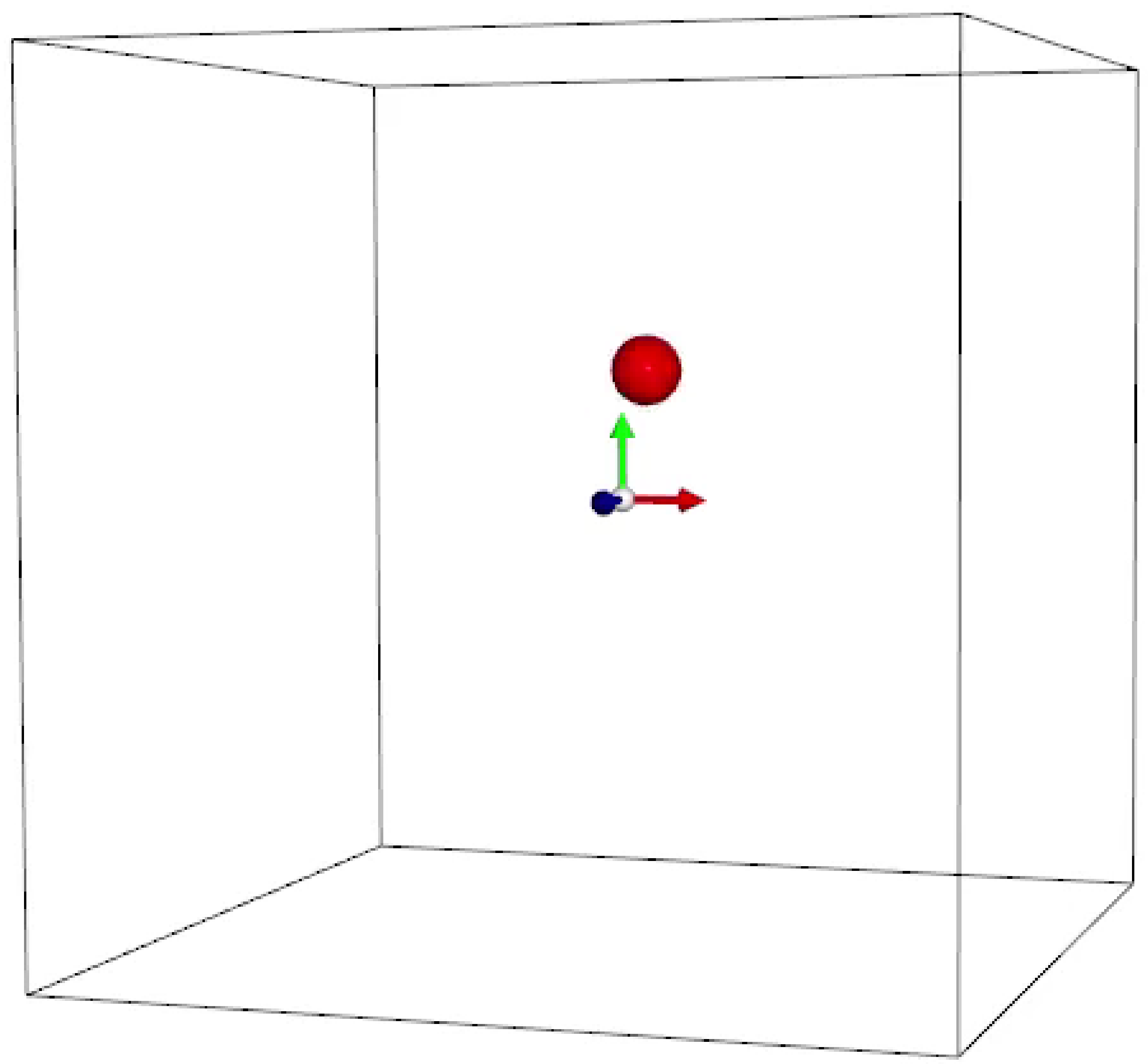
$v^{new} = \alpha\, v_{//} - \beta\, v_{\perp}$

- $\alpha \in [0, 1]$ Restitution coefficient in $//$ direction (friction)
- $\beta \in [0, 1]$ Restitution coefficient in $\perp$ direction (impact)



Result: Collision response

Applying collision response on speed only

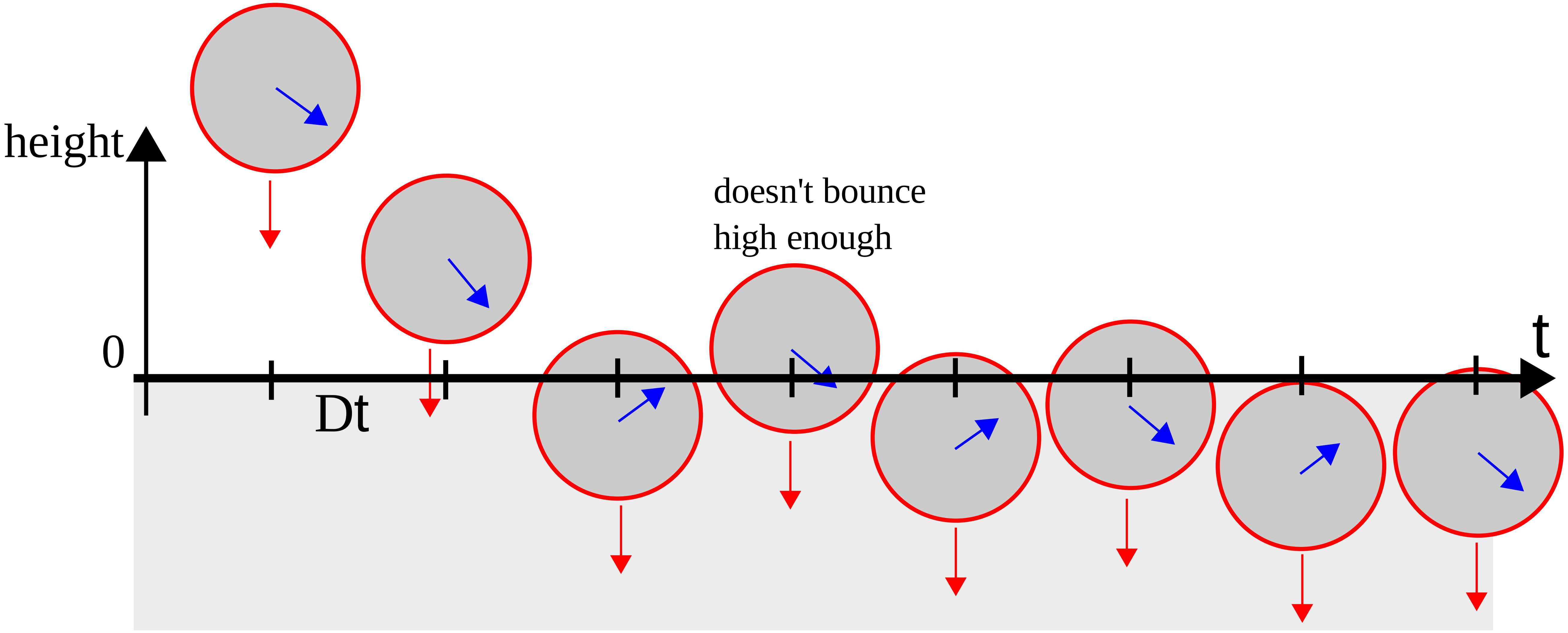


Result: Collision response - issue with discrete time

We assumed contact b/w sphere and plane

But: Exact contact never happens in discrete time

- *When collision is detected* $\rightarrow$ *already inside the wall*
- *Weight is still acting*



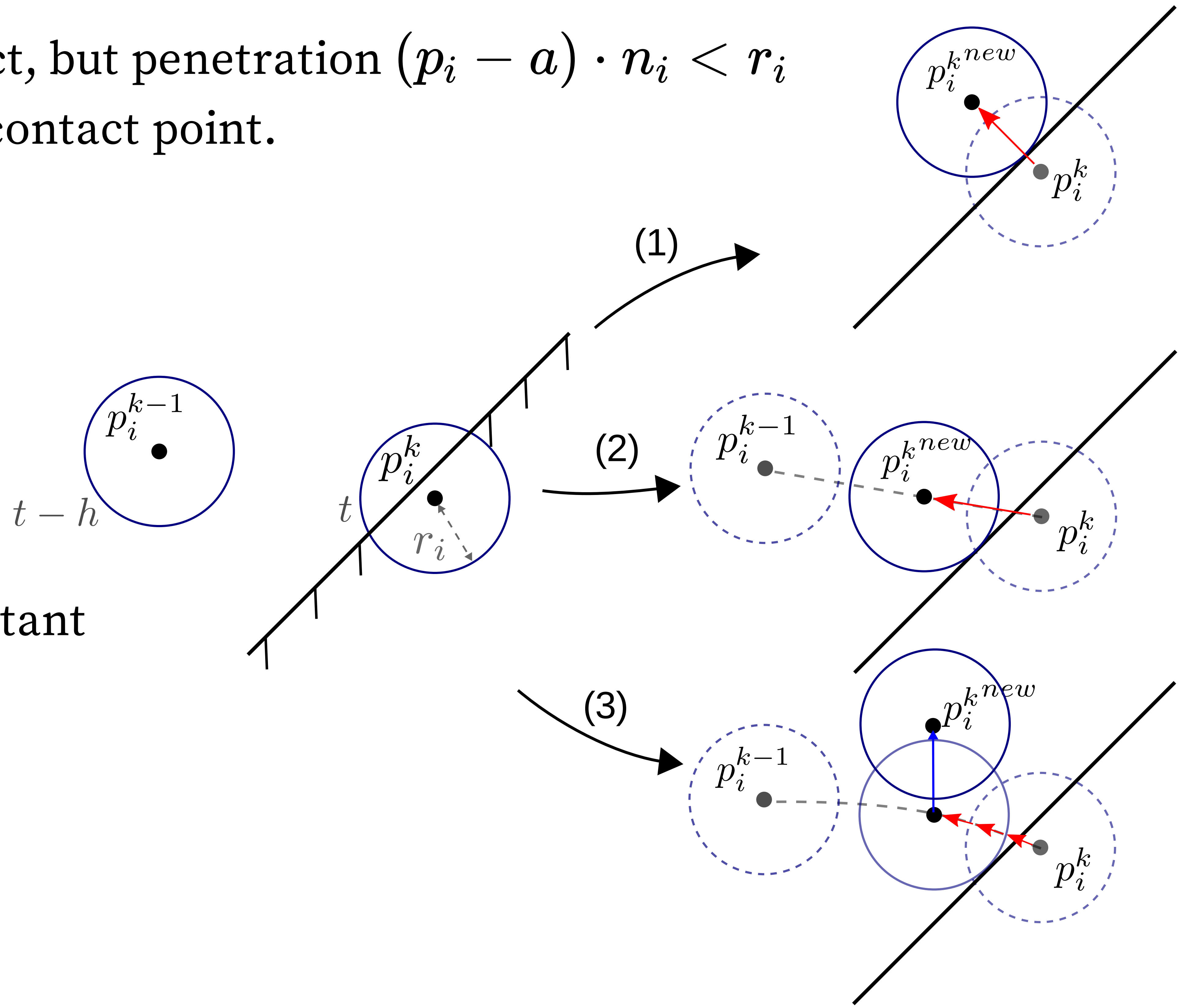
Collision response with plane : position

In real case (discrete time) no exact contact, but penetration $(p_i - a) \cdot n_i < r_i$

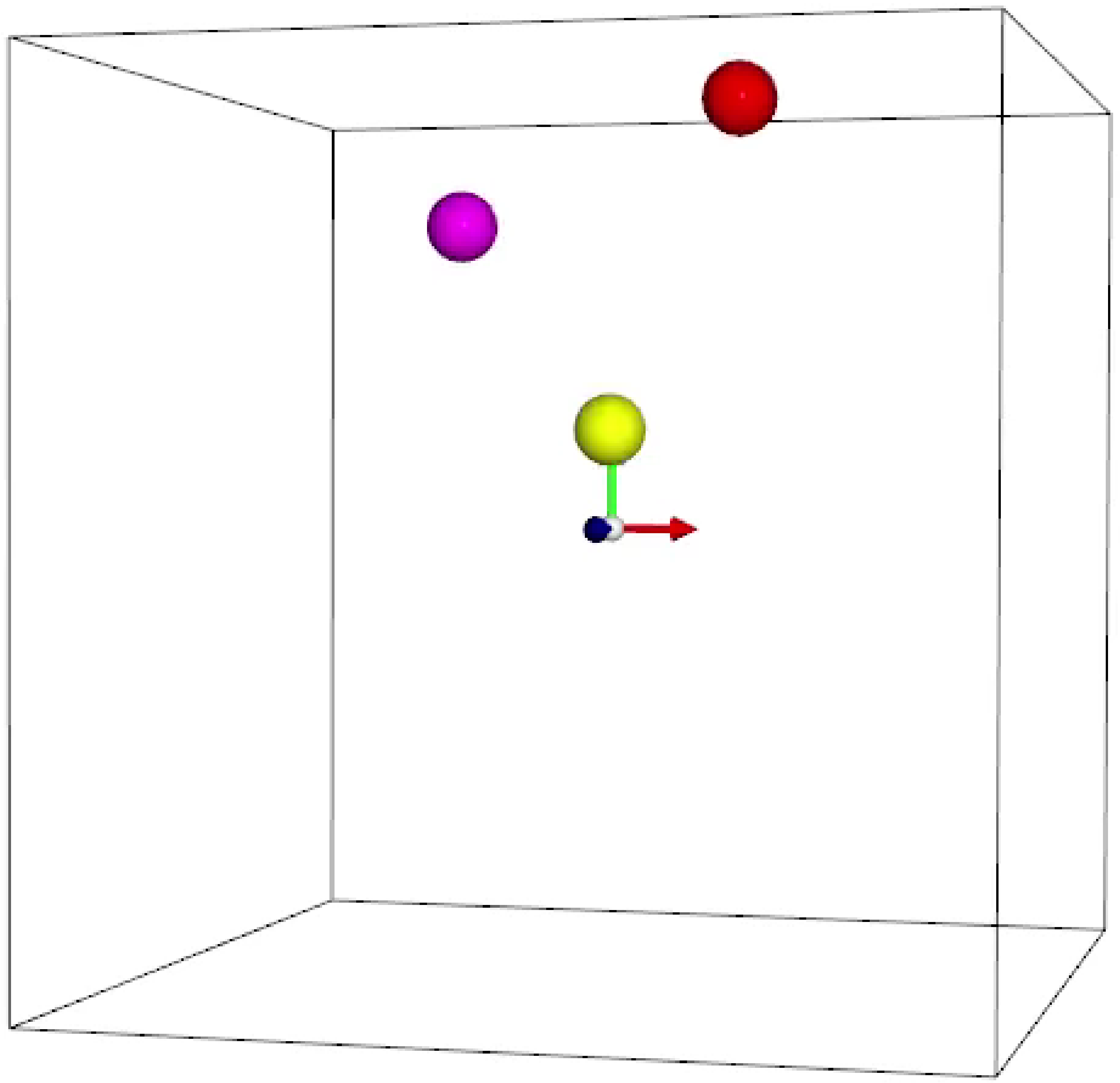
$\Rightarrow$ Need to compute collision response at contact point.

Three possibilities

- (1) Correct position in projecting on the constaint
  - *(+) Simple to implement*
  - *(-) Physically incorrect position*
- (2) Approximate the correct position
- (3) Go backward in time to find exact instant of collision
  - *Continuous Collision Detection*
  - *(+) Physically correct*
  - *(-) Computationally heavy (binary search, etc.)*



Result: Projecting position on plane



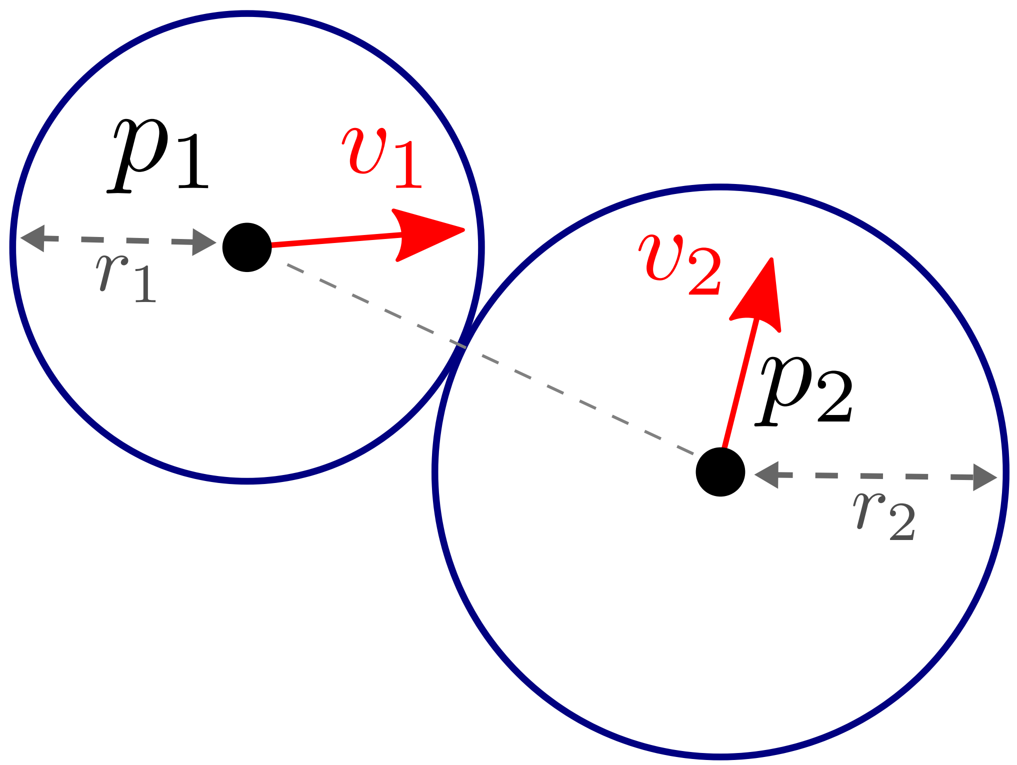
$p_i^{new} = p_i + d\, n$

$d = r_i - (p_i - a) \cdot n_i$: distance of penetration

Collision between spheres

Given 2 spheres (p_1, v_1, r_1, m_1), (p_2, v_2, r_2, m_2).

Collision when $\|p_1 - p_2\| \leq r_1 + r_2$



What happen with their velocities ?

$v_1 \rightarrow v_1^{new}, v_2 \rightarrow v_2^{new}$

Notion of impulse

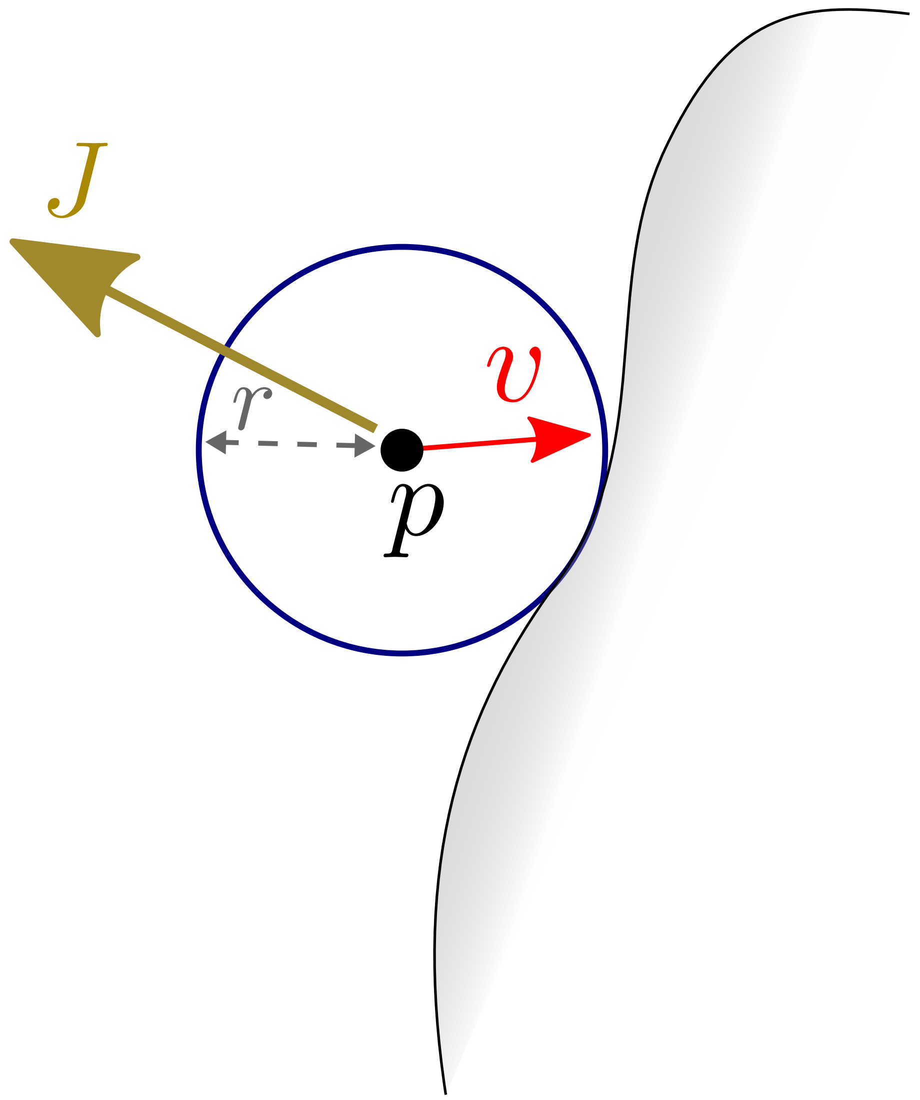
An impulse J is the integrated force over time $J = \int_{t_1}^{t_2} F(t)\,\mathrm{d}t$

$\rightarrow$ results in a sudden change of speed (/momentum) in a discrete case

For a particle with constant mass

$$\int_{t_1}^{t_2} F(t)\,\mathrm{d}t = \int_{t_1}^{t_2} m\,a(t)\;\mathrm{d}t$$

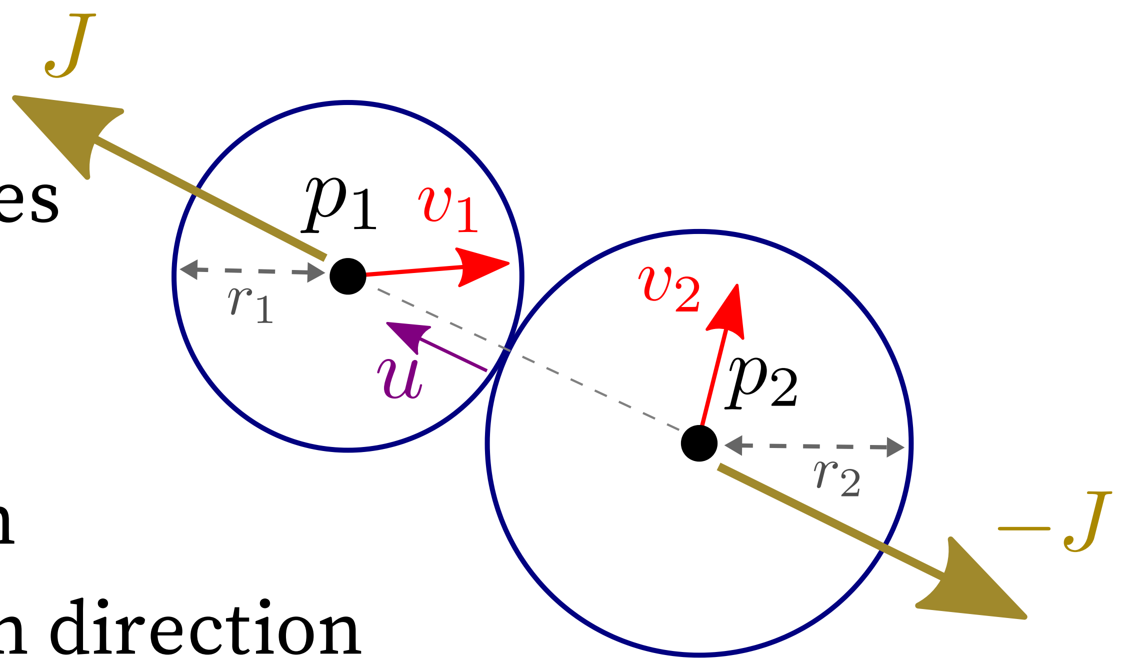
$$\Rightarrow J = m\,(v(t_2) - v(t_1))$$



For an impact $v \rightarrow v^{new}$

$$v^{new} = v + J/m$$

Two spheres in collision



Impulse orthogonal to the separating plane between the two surfaces

$$J = j\,u, \quad u = (p_1 - p_2)/\|p_1 - p_2\|$$

The system with the two spheres is preserving its linear momentum

$\Rightarrow$ Respective impulses j are equals in magnitude, and opposed in direction

$$m_1 v_1 + m_2 v_2 = m_1 v_1^{new} + m_2 v_2^{new} \Rightarrow m_1(v_1^{new} - v_1) = -m_2(v_2^{new} - v_2) \Rightarrow J_1 = -J_2$$

Assume collision of "hard spheres" = "Elastic collision"

= No loss of energy, conservation of kinetic energy of the system

$$\Rightarrow j = 2\frac{m_1\, m_2}{m_1 + m_2}\,(v_2 - v_1)\cdot u$$

$$1/2\, m_1\, v_1^2 + 1/2\, m_2\, v_2^2 = 1/2\, m_1\,(v_1^{new})^2 + 1/2\, m_2\,(v_2^{new})^2$$

$$\Rightarrow m_1 v_1^2 + m_2 v_2^2 = m_1\left(v_1 + \frac{j}{m_1}u\right)^2 + m_2\left(v_2 - \frac{j}{m_2}u\right)^2$$

$$\Rightarrow 0 = 2\,j\; v_1\cdot u + \frac{j^2}{m_1} - 2\,j\; v_2\cdot u + \frac{j^2}{m_2}$$

$$\Rightarrow j = \frac{2}{1/m_1 + 1/m_2}\,(v_2 - v_1)\cdot u$$

Two spheres in collision

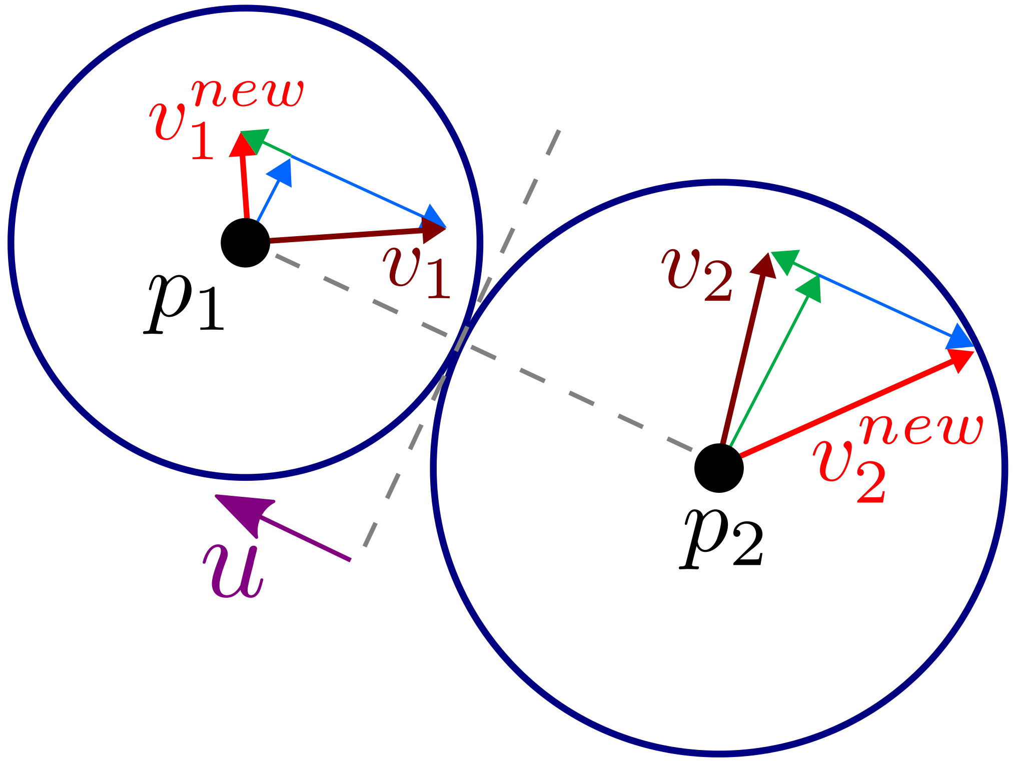
$$v_1^{new} = v_1 + j/m_1\, u = v_1 + 2\frac{m_2}{m_1+m_2}((v_2 - v_1)\cdot u)\, u$$
$$v_2^{new} = v_2 - j/m_2\, u = v_2 - 2\frac{m_1}{m_1+m_2}((v_2 - v_1)\cdot u)\, u$$

Rem. If $m_1 = m_2$: Switch their $\perp$ speeds

$$v_1^{new} = v_1 + ((v_2 - v_1)\cdot u)\, u = v_{1//} + v_{2\perp}$$
$$v_2^{new} = v_2 - ((v_2 - v_1)\cdot u)\, u = v_{2//} + v_{1\perp}$$

Can use restitution coefficient and attenuation $(\alpha, \beta) \in [0, 1]$

$$v_1^{new} = \alpha\, v_{1//} + \beta\, v_{2\perp}$$
$$v_2^{new} = \alpha\, v_{2//} + \beta\, v_{1\perp}$$



Summary

1. Detect collision $\|p_1 - p_2\| \leq r_1 + r_2$

2-a. If collision (relative speed $> \epsilon$)

Elastic collision (/bouncing) $v_{1/2} = \alpha\, v_{1/2} \pm \beta\, J / m_{1/2}$

2-b. If *static* contact (relative speed $\leq \epsilon$)

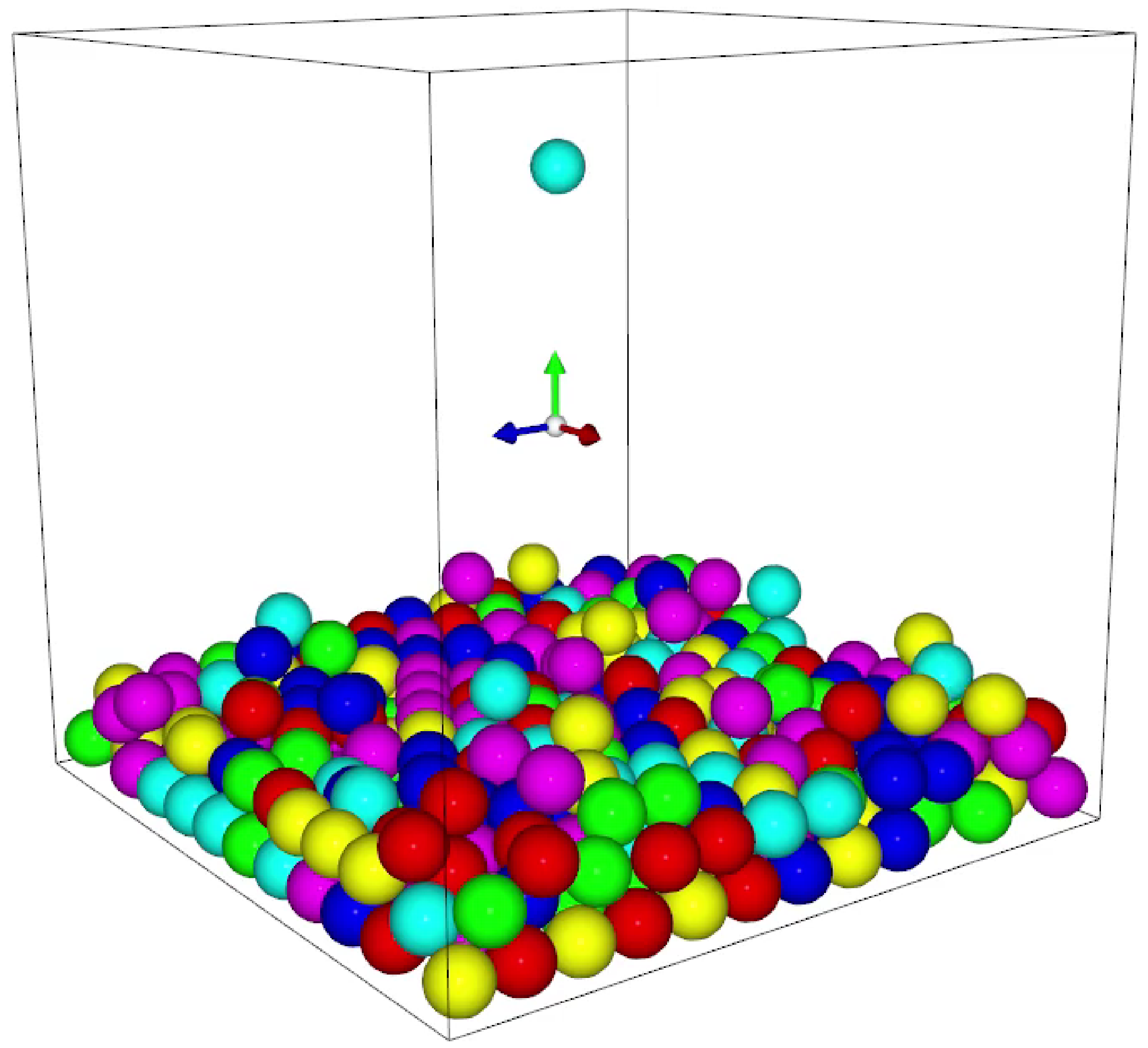
Friction $v_{1/2} = \mu\; v_{1/2}\,, \quad \mu \in [0, 1]$

Avoids jittering

3. Correct position (project on contact surface)

$p = p \pm d/2\, u$

$d = r_1 + r_2 - \|p_1 - p_2\|$: Collision depth

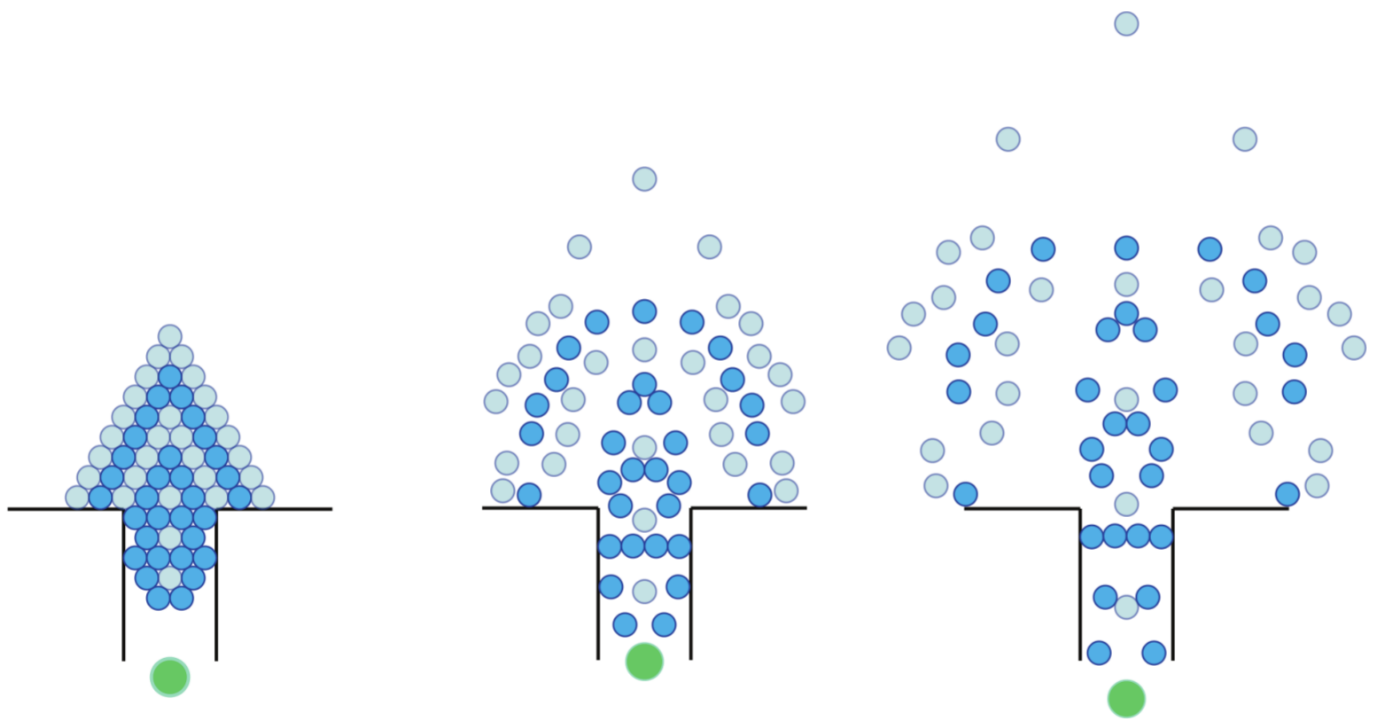


Multiple collisions

Pairwise collisions $\Rightarrow$ no global collision free state

- Correcting one collision may induce new collisions.
- Order of correction does matter

Reducing time-step help, Iterating over mutliple pass help



But correct solution in all cases is complex $\rightarrow$ global approach

- Precompute contact graph

  explicit shock propagation management

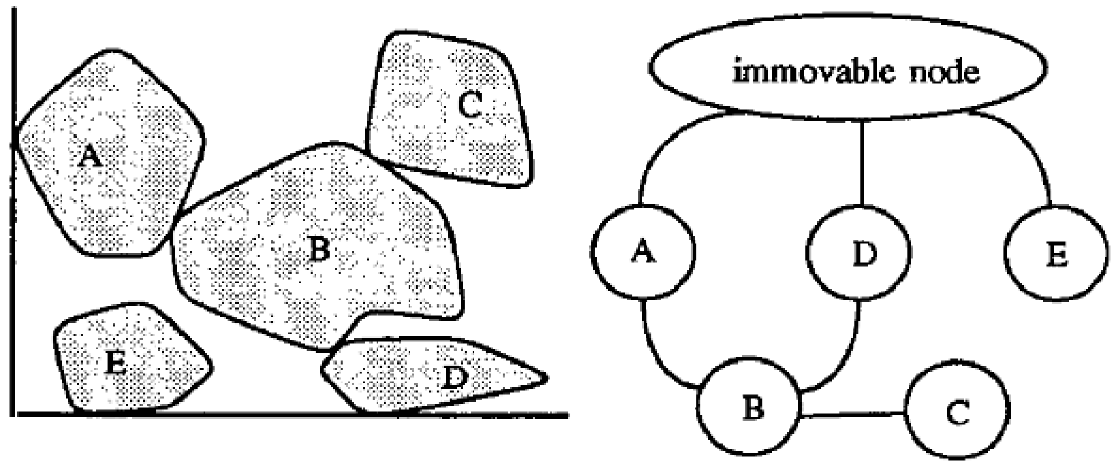
- Global constraint-based method

  Impulse: $n_i \cdot (v_i - v_j) \geq 0$

  Momentum preservation: $m_i v_i - m_j v_j = 0$

  Energy preservation/dissipation

  $\Rightarrow$ Linear Complementarity Program, Gauss Seidel, etc.



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