Animating fluids (I)

Stable Fluid

Solving Navier-Stokes on grid

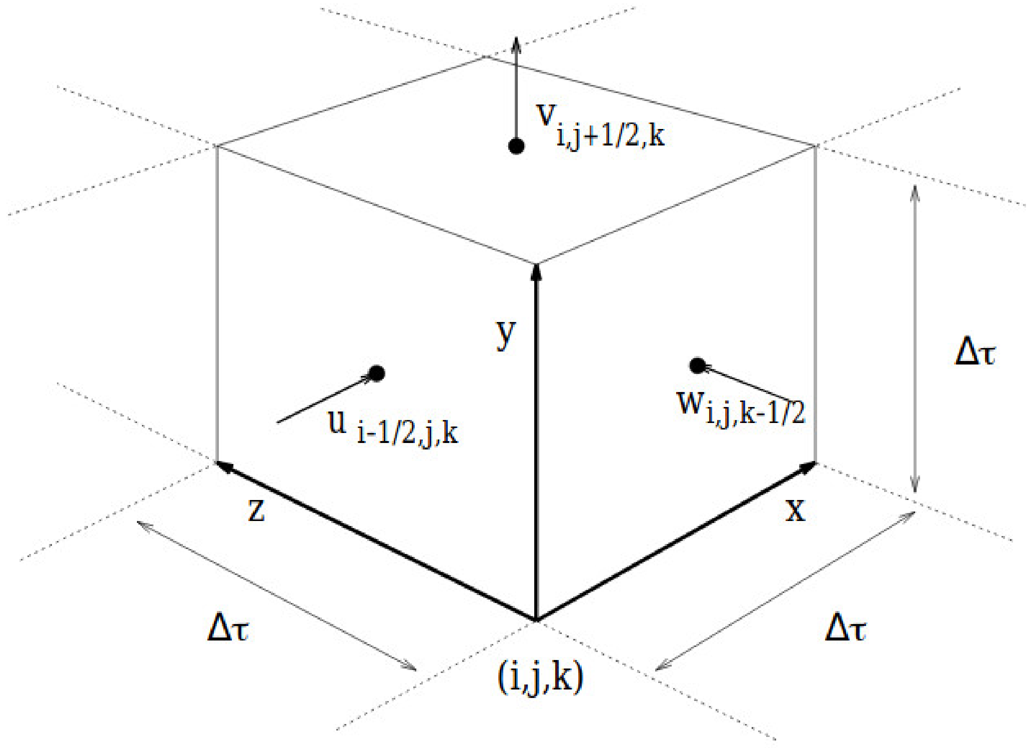
"Brute force" approach

- Rectangular grid filled with fluid
- Use finite differences on the grid for Navier-Stokes equation

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\nabla \mathrm{p} + f - (u \cdot \nabla)u + \nu \triangle u$$
$$div(u) = 0$$

(-) Stability conditions

(-) Loose advection details on the grid



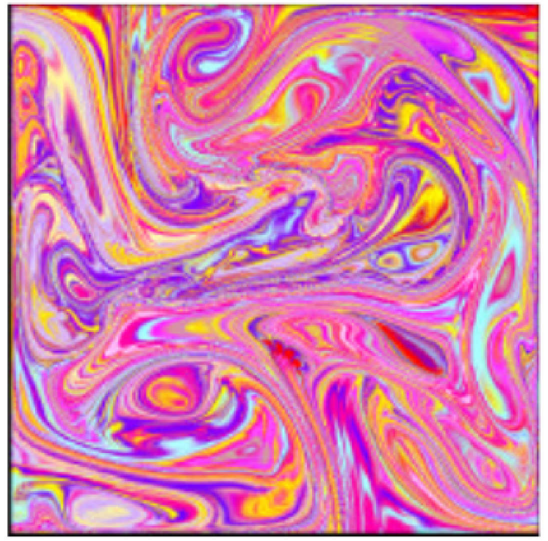
[Modeling the Motion of a Hot, Turbulent Gas. N Foster and D. Metaxas. SIGGRAPH 1997]

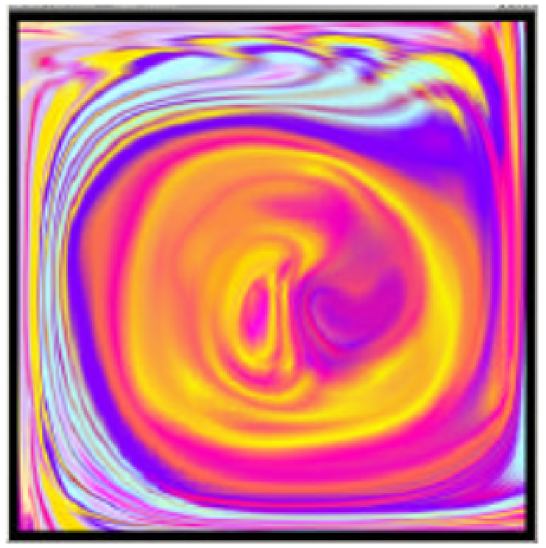
Stable Fluids - Idea

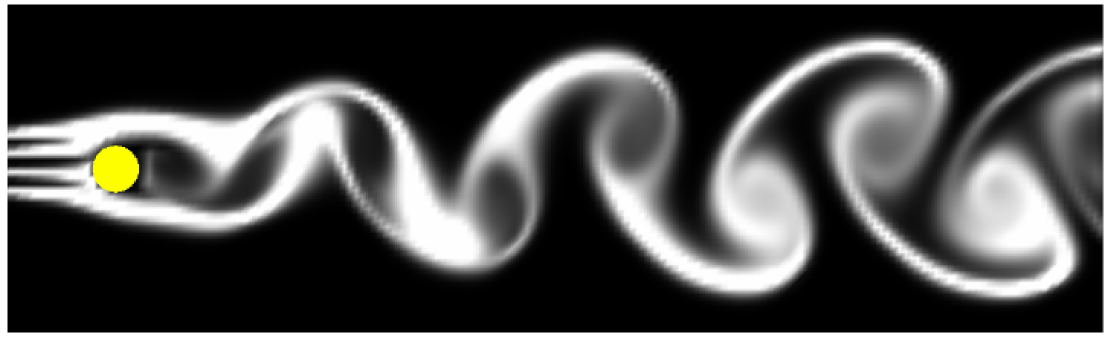
Well known improvement: **Jos Stam, Stable Fluids, ACM SIGGRAPH 1999**

$$\frac{\partial u}{\partial t} = f - (u \cdot \nabla)u + \nu \triangle u - \frac{1}{\rho}\nabla \mathrm{p}$$

- $1/\rho \, \nabla \mathrm{p}$: Pressure term only used to ensure divergence free
- Similar to Lagrange multipler for constraints







[Stable Fluids. J. Stam. SIGGRAPH 1999]

[Real Time Fluid Dynamics for Games. J. Stam. Game Dev. Conf. 2003]

1st Idea

Remove pressure term

Replace by explicit projection on divergence free vector field P

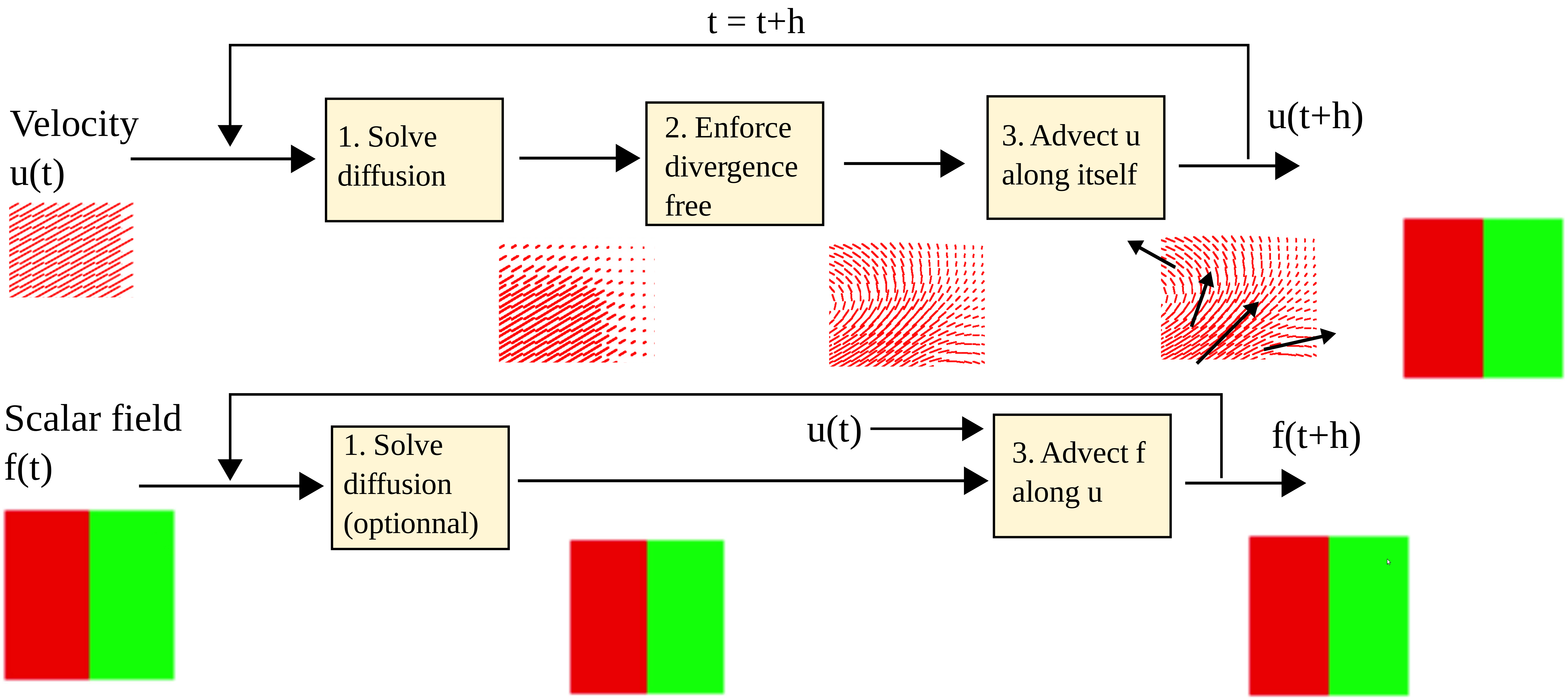
$$\Rightarrow \frac{\partial u}{\partial t} = \mathrm{P}(f - (u \cdot \nabla)u + \nu \triangle u)$$

2nd Idea

Compute each terms one after the other

$$u^k \underbrace{\rightarrow}_{\substack{add forces \\ f}} u_1^k \underbrace{\rightarrow}_{\substack{diffuse \\ \nu\triangle u}} u_2^k \underbrace{\rightarrow}_{\substack{project \\ \mathrm{P}}} u_3^k \underbrace{\rightarrow}_{\substack{advect \\ (u\cdot\nabla)u}} u^{k+1}$$

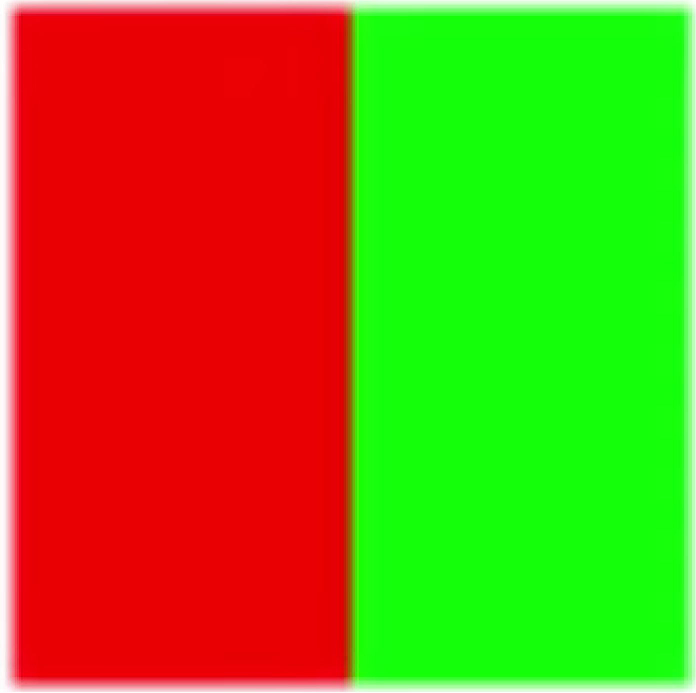
Stable Fluids - General Algorithm



1 - Diffusion

Use finite difference on $\dfrac{\partial f}{\partial t} = \nu \triangle f$

Notation: $f^t_{x,y} = f(k_x\,\Delta x,\ k_y\,\Delta y,\ k_t\,\Delta t)$



Explicit schemes may oscillates/diverge for large time steps

$\Rightarrow$ Use implicit scheme for unconditional stability

$$\frac{f^{k+1}_{x,y} - f^{k}_{x,y}}{\Delta t} = \nu\left(\frac{f^{k+1}_{x+1,y} - 2\,f^{k+1}_{x,y} + f^{k+1}_{x-1,y}}{(\Delta x)^2} + \frac{f^{k+1}_{x,y+1} - 2\,f^{k+1}_{x,y} + f^{k+1}_{x,y+1}}{(\Delta y)^2}\right)$$

Assuming $\Delta x = \Delta y = 1$

$$(1 + 4\,\nu\Delta t)\,f^{k+1}_{x,y} - \nu\,\Delta t(f^{k+1}_{x+1,y} + f^{k+1}_{x-1,y} + f^{k+1}_{x,y+1} + f^{k+1}_{x,y-1}) = f^{k}_{x,y}$$

Use Gauss-Seidel iterative method to solve the sparse linear system

Initialize $f^{k+1} = f^k$

for $i = 1..N_{\text{max}}$

$$f^{k+1}_{x,y} = \tfrac{1}{1+4\,a}(f^{k}_{x,y} + a(f^{k+1}_{x-1,y} + f^{k+1}_{x+1,y} + f^{k+1}_{x,y-1} + f^{k+1}_{x,y+1}))\ ,\quad a = \nu\,\Delta t$$

2 - Advection

Advection = move some function along given velocity u.

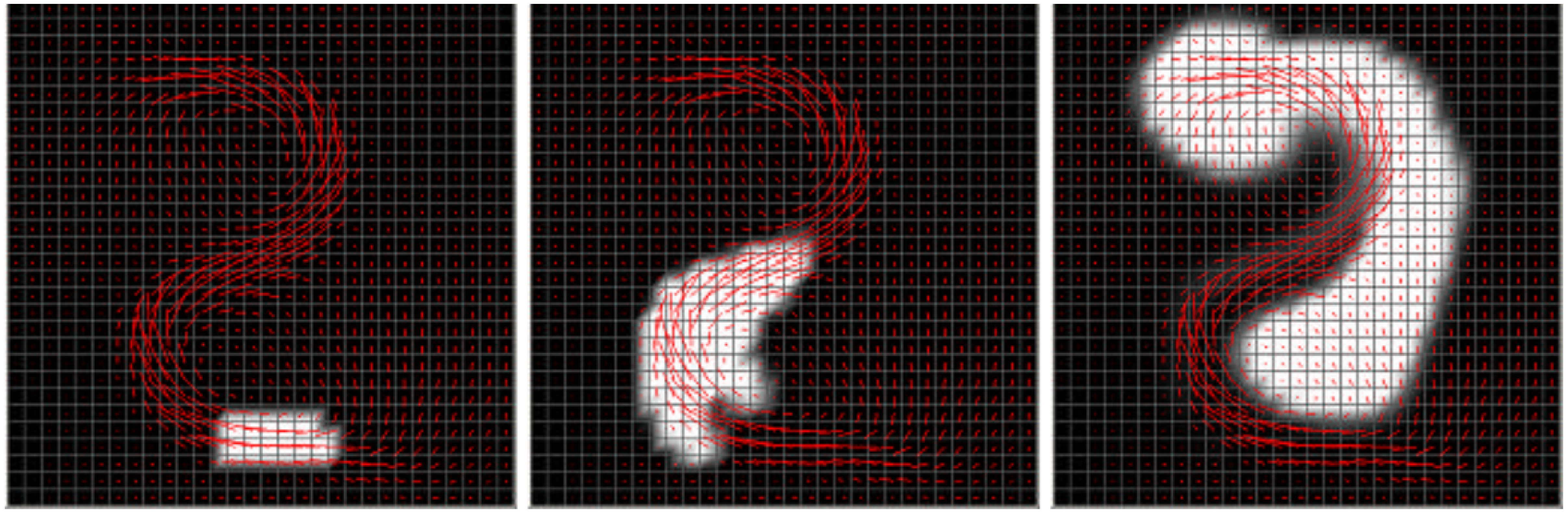
- Advecting a scalar field f along u

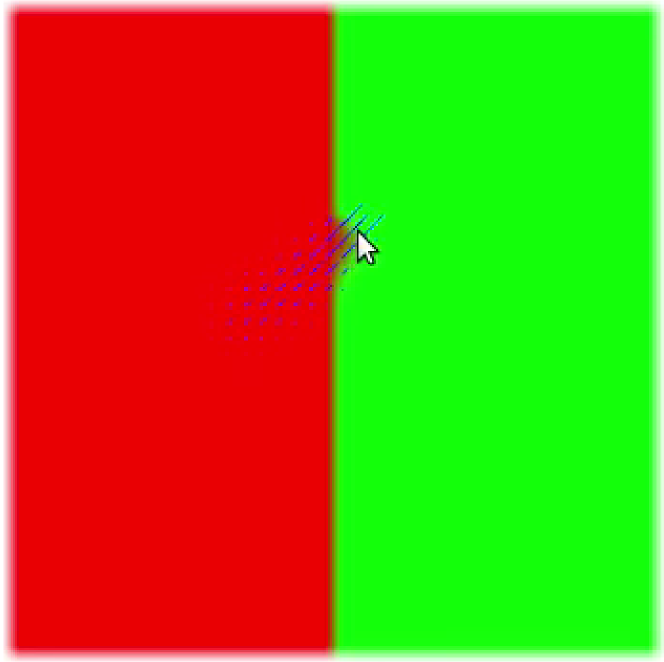
$$\frac{\partial f}{\partial t}(p,t) + u(p,t) \cdot \nabla f = 0$$

- Advecting a vector field f along u

$$\frac{\partial f}{\partial t}(p,t) + (u(p,t) \cdot \nabla)\, f = 0$$

- In Navier-Stokes advect the velocity itself $f = u$
- Can also advect density, color, texture coordinates, etc. to visualize the motion.





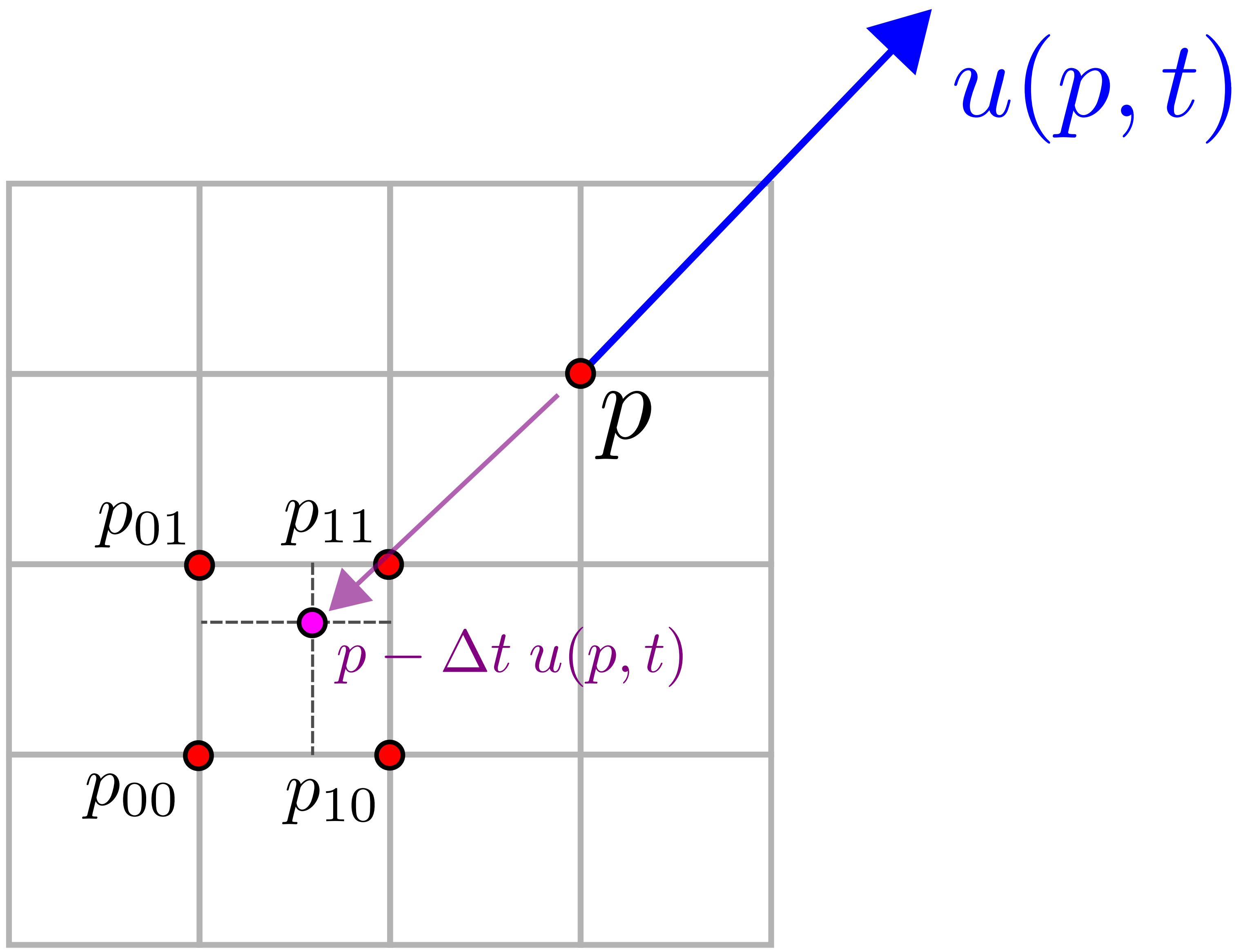
2 - Computing advection

Advecting generic value f along u

Idea Compute value of f at time t at fixed position grid p in moving back at $t - \Delta t$.

Value of f advected at point p at time t was at position $p_{prev} = p - \Delta t\ v(p,t)$ at time $t - \Delta t$.

$$\Rightarrow f(p,t) = f(p_{prev}, t - \Delta t)$$

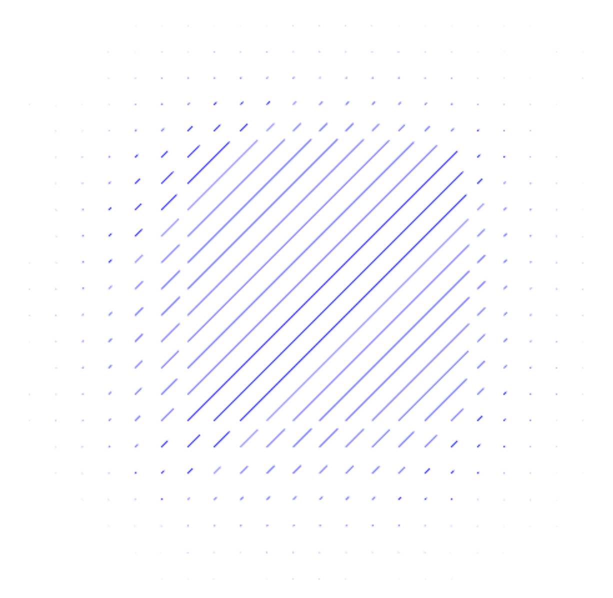


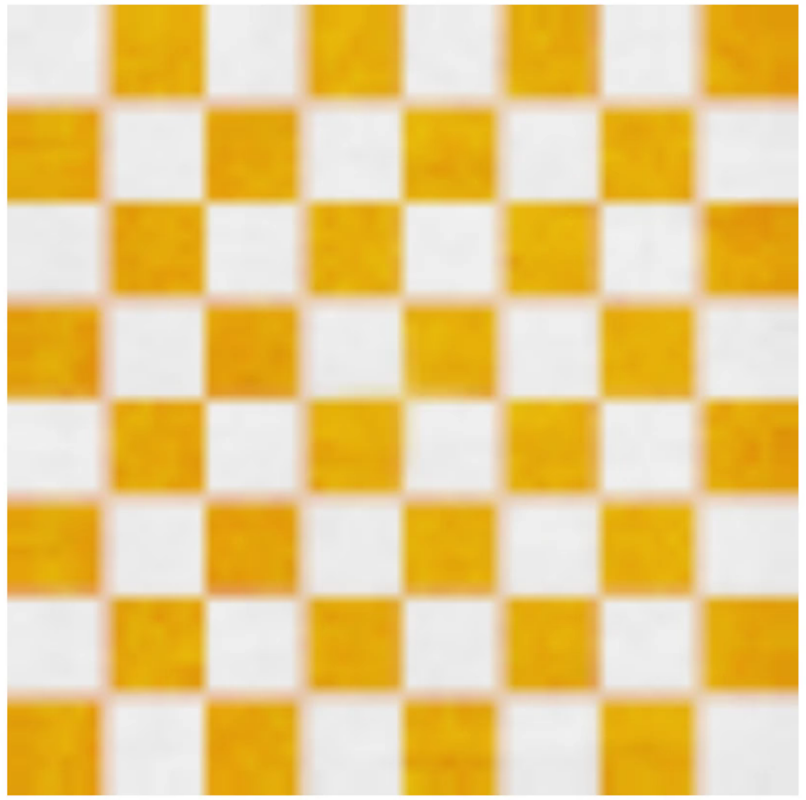
p_{prev} is not a grid point coordinates: Use interpolation
Can use Bilinear interpolation

$$f(p_{prev}) = (1-\alpha)(1-\beta)\ p_{00} + (1-\alpha)\,\beta\, p_{01} + \alpha\,(1-\beta)\ p_{10} + \alpha\,\beta\ p_{11}$$

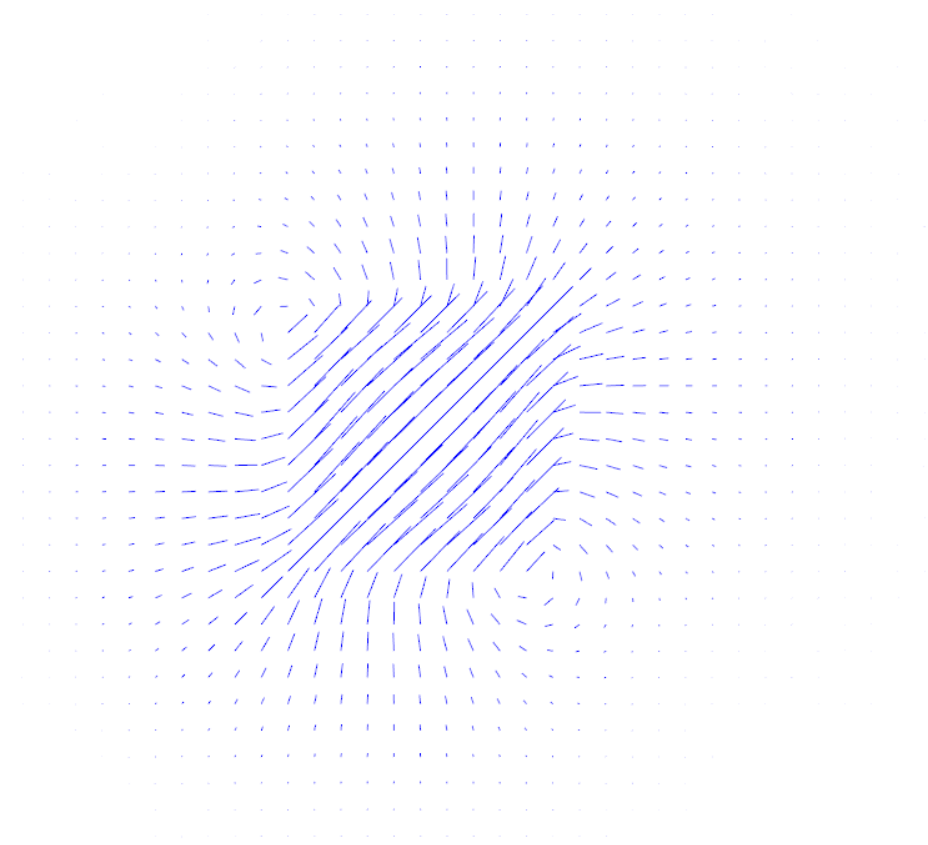
3 - Divergence Free Vector Field

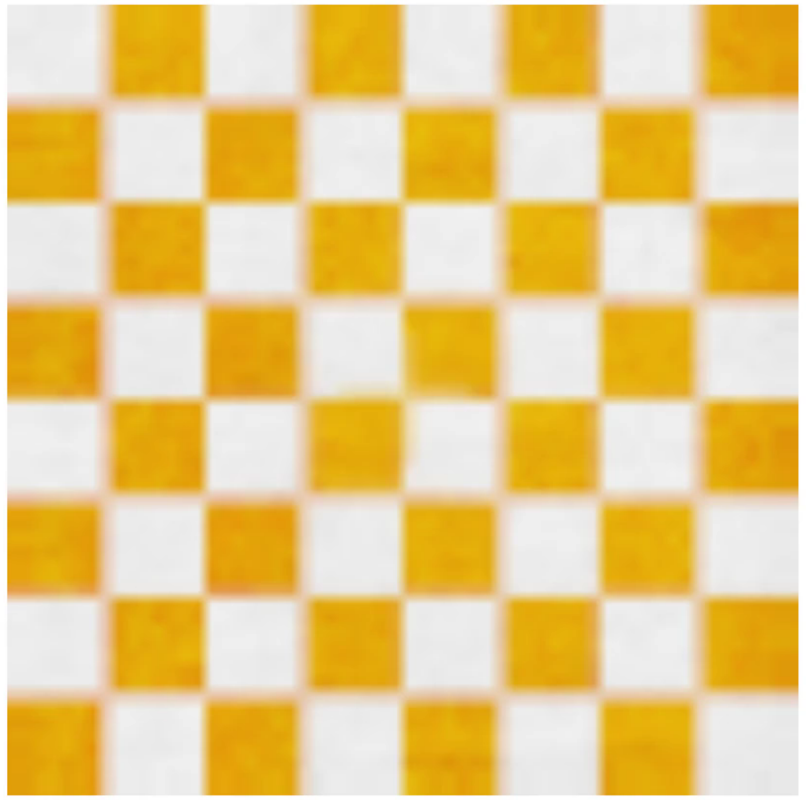
Before projection:





After projection:





$$div(v) = 0$$

$$v^x_{x+1,y} + v^x_{x-1,y} + v^y_{x,y+1} + v^y_{x,y-1} = 0$$

3 - Projection to divergence free vector field

Consider a general vector field w

Helmoltz decomposition: $w = u + v$

- u: Divergence free vector field such that $div(u) = 0$
- v: Gradient field $v = \nabla q$, q scalar field.

q satisfies a Poisson equation

$$div(w) = \underbrace{div(u)}_{=0} + div(v) \Rightarrow div(w) = \underbrace{div(\nabla q)}_{\triangle q}$$

Method- Given an input field w

1. Compute q as solution of $\triangle q = div(w)$
2. Compute $u = w - \nabla q$

3 - Projection to divergence free vector field (Algo)

Input vector field $w = (w^x, w^y)$

Note: we assume in the following $\Delta x = \Delta y = 1$

1 - Compute $d = \mathrm{div}(w)$

$$d_{x,y} = (w^x_{x+1,y} - w^x_{x-1,y} + w^y_{x,y+1} - w^y_{x,y-1})/2$$

2 - Compute q in solving $\triangle q = b$

$$(q_{x+1,y} + q_{x-1,y} - 2\, q_{x,y}) + (q_{x,y+1} + q_{x,y-1} - 2\, q_{x,y}) = d_{x,y}$$
$$\Rightarrow 4q_{x,y} = q_{x+1,y} + q_{x-1,y} + q_{x,y+1} + q_{x,y-1} - d_{x,y}$$

ex. Numerical iterations using Gauss Seidel

Initialize $q = 0$

For $i = [1..N_{\mathrm{max}}]$

$$q_{x,y} = 1/4\,(q_{x+1,y} + q_{x-1,y} + q_{x,y+1} + q_{x,y-1} - d_{x,y})$$

3 - Compute $u = w - \nabla q$

$$u_{x,y} = w_{x,y} - (q_{x+1,y} - q_{x-1,y}\ ,\ q_{x,y+1} - q_{x,y-1})\,/2$$

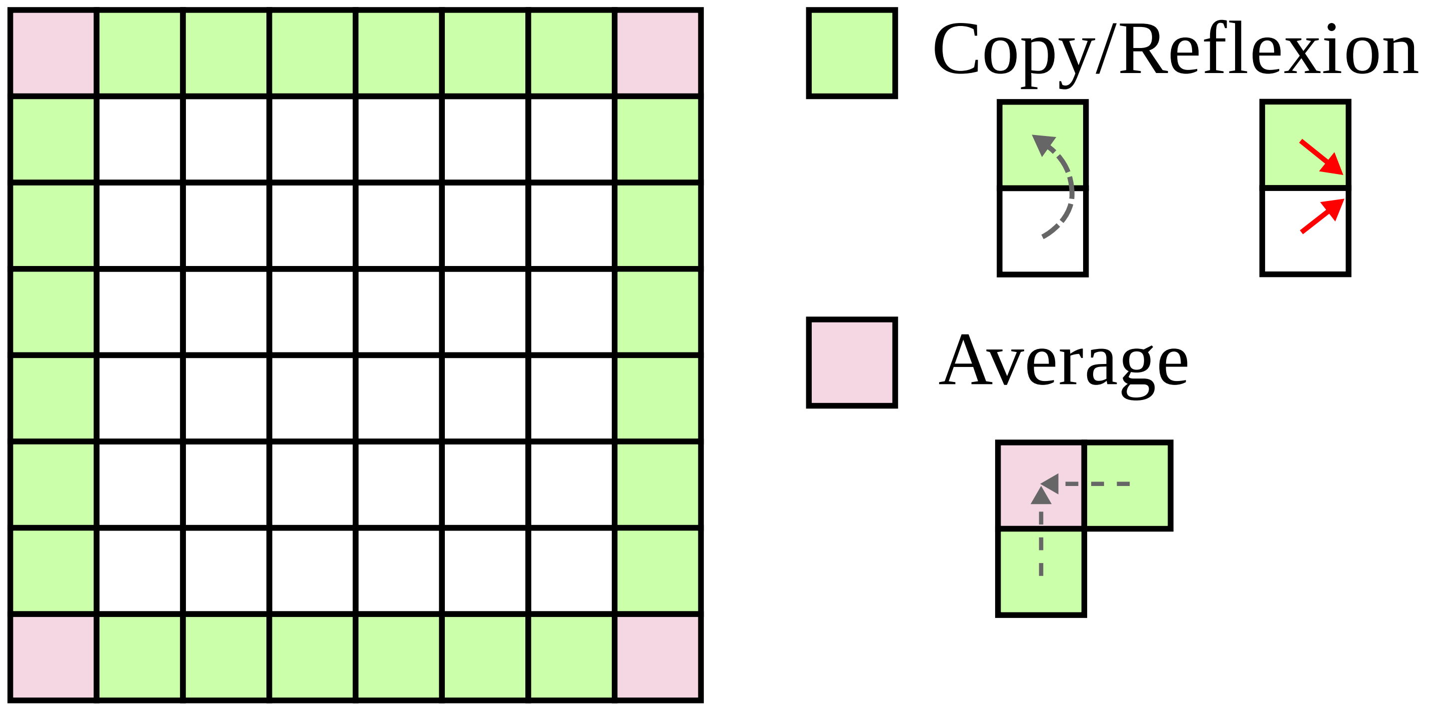
Handling boundaries

Boundaries $x = 0, x = N_x - 1, y = 0, y = N_y - 1$
need special care

- For density
  Assume value C^0 continuity on the boundary
  Row/Column $f_{x,0} = f_{x,1}, f_{0,y} = f_{1,y}$ etc.

- For velocity: $f = (f^x, f^y)$
  Assume reflexion on walls
  Row: $f_{x,0} = (f^x_{x,1}, -f^y_{x,1})$
  Column: $f_{0,y} = (-f^x_{1,y}, f^y_{1,y})$

- In all cases: Average value for corners
  $f_{0,0} = (f_{1,0} + f_{0,1})/2$, etc.



Stable fluids example

?