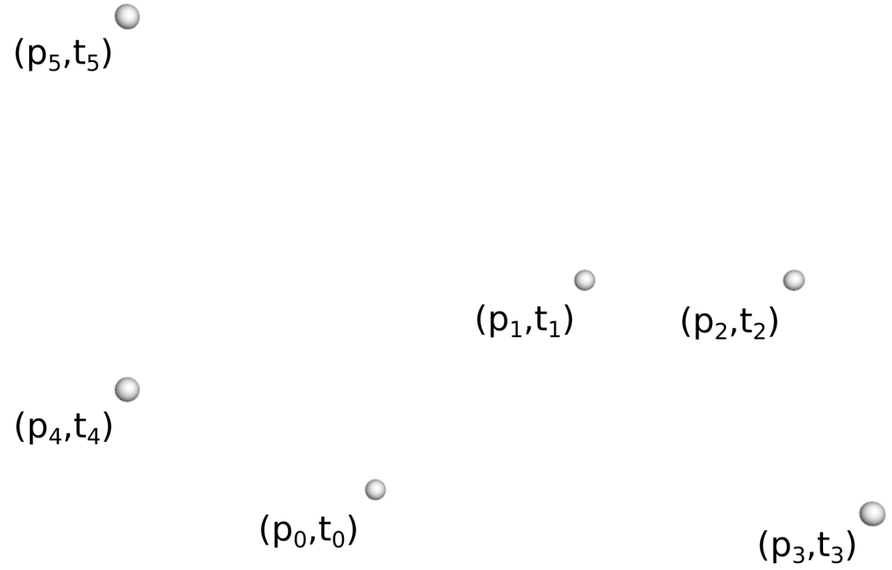
Interpolating positions

Spline trajectory

Objective

- Given a set of key-positions (positions+time) we want to find an interpolating space-time curve



- **Input** $(p_i, t_i), i \in [0, N-1]$
- **Output**
  - Space time curve $p(t), t \in [t_0, t_{N-1}]$
  - Space time curve $p(t_i) = p_i$

Linear Interpolation

- Simplest solution: linear interpolation between each sample pairs (p_i, p_{i+1})

- $\forall t \in [t_i, t_{i+1}]$, $\begin{cases} p(t) = (1 - \alpha(t))\, p_i + \alpha(t)\, p_{i+1} \\ \alpha(t) = \dfrac{t - t_i}{t_{i+1} - t_i} \end{cases}$

Pros

- Simple
- Constant speed between keyframes

  Easy to ajust

Cons

- Non smooth trajectory
- Generates straight segments

  Artists prefer non straight trajectories for "real" looking motion

Smooth curve

Objective: Generate a **smooth** interpolating space-time curve

Classical Idea Use polynomials curves

$$\begin{cases} p(t) = \sum_{i=0}^{N-1} \alpha_i(t)\, p_i \\ \alpha_i(t) = \sum_{j=0}^{d} c_i^j\, t^i \end{cases}$$

(α_i) polynomial basis function of degree d

Which polynomials/degree choose ?

Lagrange polynomial interpolation

Naive idea: Interpolate all points at once

$$\forall t \in [t_0, t_{N-1}]\ ,\quad p(t) = \sum_{i=0}^{N-1} \alpha_i(t)\ p_i \qquad \forall i \in [0, N-1]\ \ p(t_i) = p_i$$

Degree of polynomial : $N-1$

- Known solution: **Lagrange polynomial**

$$p(t) = \sum_{i=0}^{N-1} \alpha_i(t)\ p_i \qquad \alpha_i(t) = \prod_{k=0, k\neq i}^{N-1} \frac{t - t_k}{t_i - t_k}$$

- Explanation: By construction $\alpha_i(t_i) = 1$ and $\alpha_i(t_k) = 0$

(+) Interpolate all points

(-) Large oscillations between samples for large degree.

(-) Non local influence

=> Not used in practice

Lagrange polynomial interpolation : Comparison

Ex. Global effect on curve when two samples are added.

10 samples

12 samples

Spline

Idea

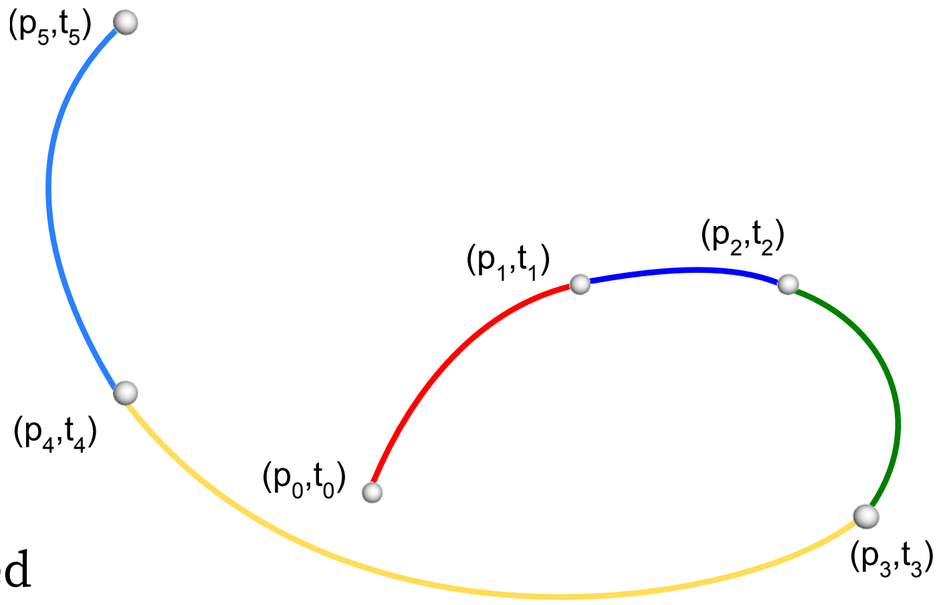
- Define on each part a polynomial
- Smooth junctions between them.

How to choose the polynomial

- Sufficiently high degree to be smooth
- Sufficiently low degree to avoid oscillations

=> In Graphics cubic polynomials are often used

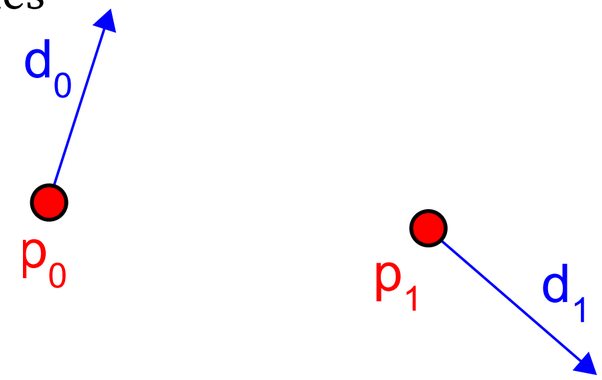
Allows up to C^2 junctions



Each color is another polynomial

Hermite interpolation

Hermite interpolation : cubic curve interpolating points and derivatives at extremities

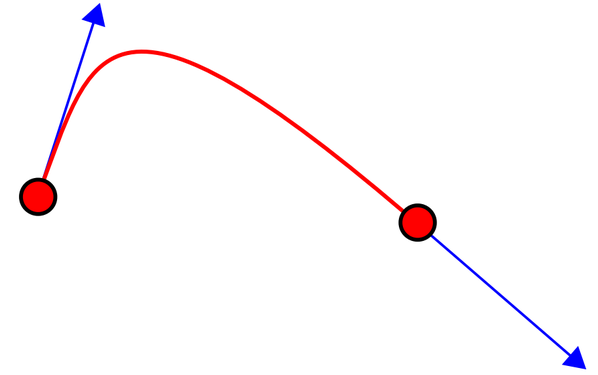


Consider the following constraints

$$\begin{cases} p(s) = c_3 s^3 + c_2 s^2 + c_1 s + c_0 \\ p(0) = p_0,\ p(1) = p_1,\ p'(0) = d_0,\ p'(1) = d_1 \end{cases}$$

$\Rightarrow$ System of equations

$$\begin{cases} & & & c_0 & = p_0 \\ c_3 + & c_2 + & c_1 + & c_0 & = p_1 \\ & & c_1 & & = d_0 \\ 3\,c_3 + & 2\,c_2 + & c_1 & & = d_1 \end{cases} \Rightarrow \begin{cases} c_0 = p_0 \\ c_1 = d_0 \\ c_2 = -3p_0 + 3p_1 - 2d_0 - d_1 \\ c_3 = 2p_0 - 2p_1 + d_0 + d_1 \end{cases}$$



$$\forall s \in [0,1]\ ,\quad p(s) = (2s^3 - 3s^2 + 1)\,p_0 + (s^3 - 2s^2 + s)\,d_0 + (-2s^3 + 3s^2)\,p_1 + (s^3 - s^2)\,d_1$$

For arbitrary $t \in [t_i, t_{i+1}]$, we set $s = \dfrac{t - t_i}{t_{i+1} - t_i}$

Wrap-up Algorithm

Compute $p(t)$ as a cubic spline interpolation

- Given keyframes $(p_i, t_i)_{i \in [0, N-1]}$
- Given time $t \in [t_1, t_{N-2}]$

1. Find i such that $t \in [t_i, t_{i+1}]$

2. Compute $d_i = \mu \dfrac{p_{i+1} - p_{i-1}}{t_{i+1} - t_{i-1}}$, and $d_{i+1} = \mu \dfrac{p_{i+2} - p_i}{t_{i+2} - t_i}$

3. Compute $p(t) = (2s^3 - 3s^2 + 1)\, p_i + (s^3 - 2s^2 + s)\, d_i + (-2s^3 + 3s^2)\, p_{i+1} + (s^3 - s^2)\, d_{i+1}$
   with $s = \dfrac{t - t_i}{t_{i+1} - t_i}$.

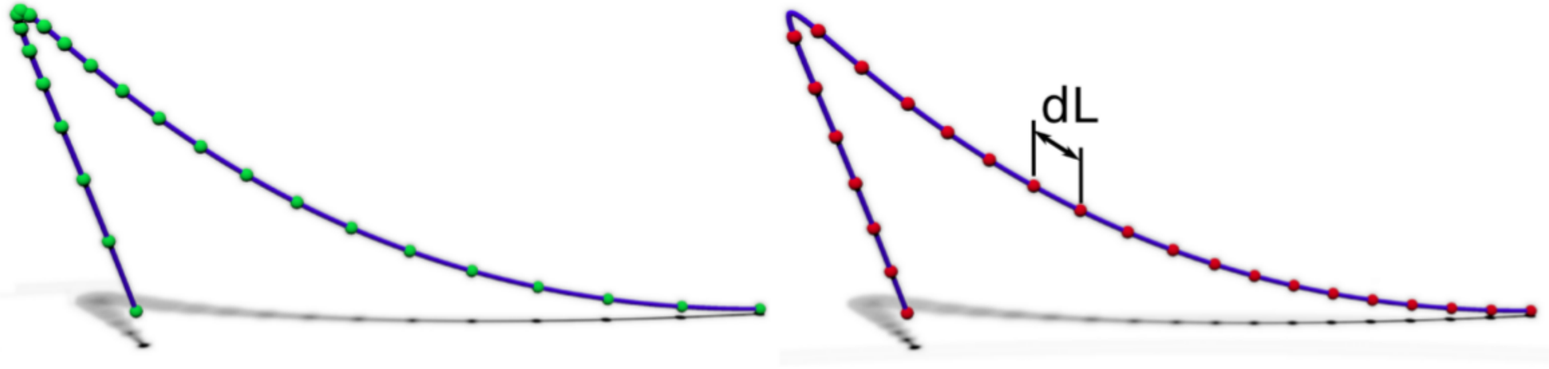
Limitation of cubic curve interpolation

- Only C^1, but not C^2 at junctions : Curvature/acceleration discontinuity

  - *Force C^2 continuity for cubic polynomial (global linear system to solve - loose local structure)*
  - *Consider higher degree polynomial*

- Non constant speed along each polynomial

  - *Reparametrization with curvilinear length*



Curve editing

- Animation software (Maya, 3DSMax, Blender, etc) always come with a **curve editor**.

- Artists can manually ajust their position, time, and **derivatives** on curve editor.
  - One curve for each scalar parameter
    - position (x, y, z)
    - scaling (s_x, s_y, s_z)
    - rotation/quaternion (q_x, q_y, q_z, q_w)

- Can also use a wrapper function w to change time $p(t) = f(w(t))$