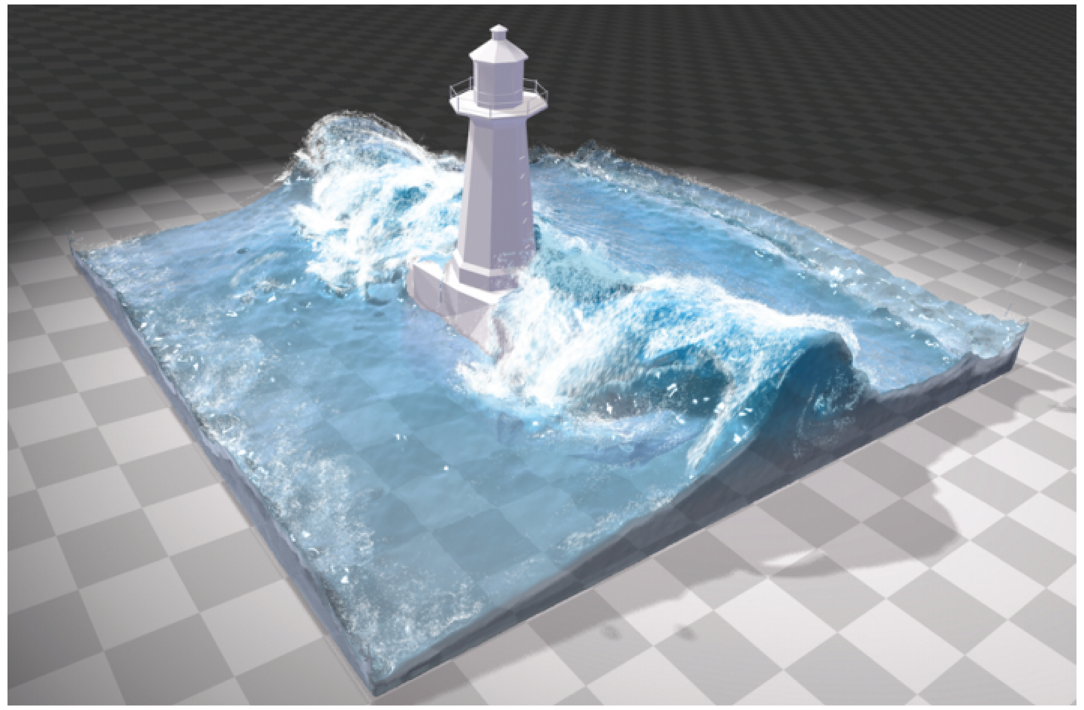
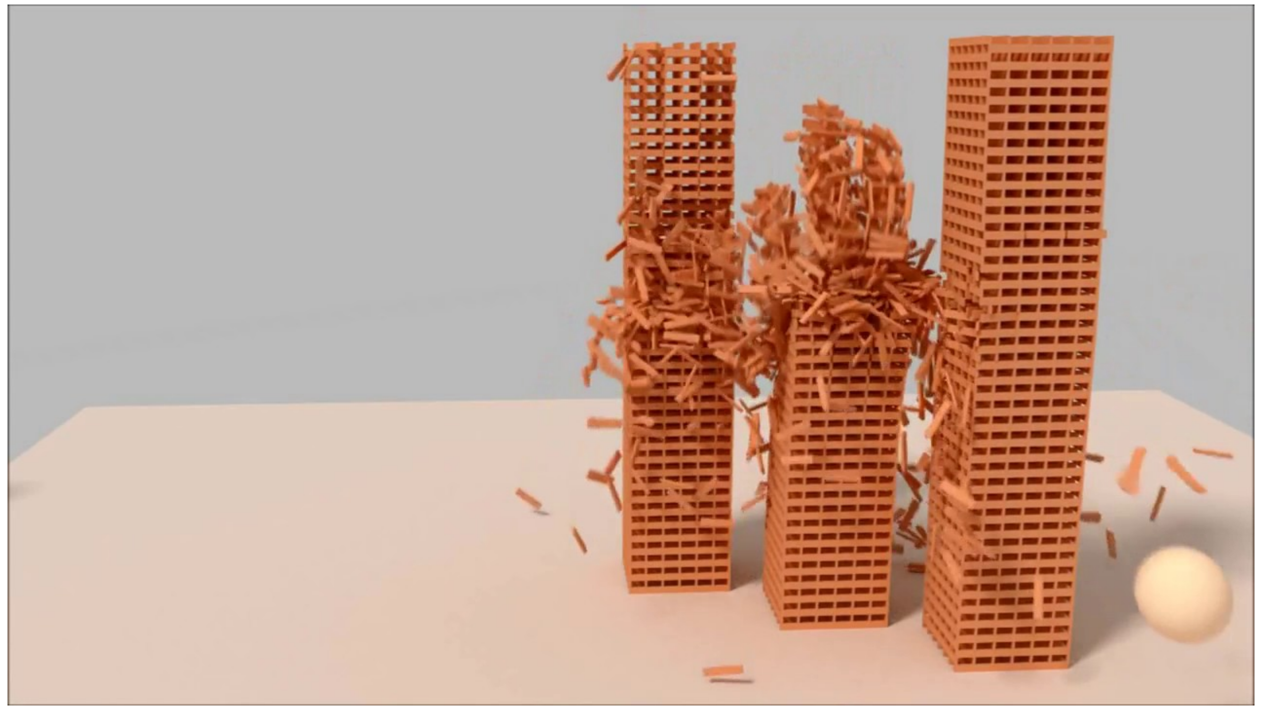
Physically based simulation - Models

When physically based simulation is needed

- Accurate dynamics
- Teadious to model by hand or procedurally
    - Multiple interacting elements: ex. Multiple collisions: rigid bodies, hairs, etc.
    - Complex animated geometry: Cloths, fluids





General methodology

1. Description of the system

Describe system by some parameters (positions, speed, orientation, etc).

- State of the system is known at time $t = 0$ - *Initial value problem in time*

- State of the system may be constrained in space - *Boundary value problem in space*

2. Evolution

Link the evolution of the system to forces or constraints using dynamic principles and conservation laws

$\Rightarrow$ Differential equation

3. Numerical Solution

Solve the differential equation using numerical iterative approaches.

Note: Fundamentally different that direct approach controling the trajectories at key-frames

- The system is set at an initial step

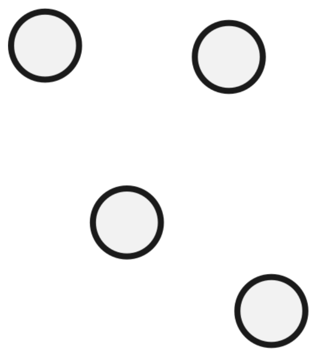
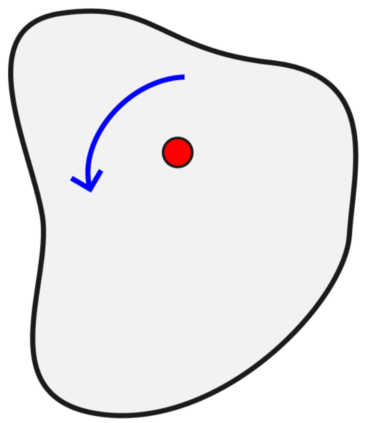
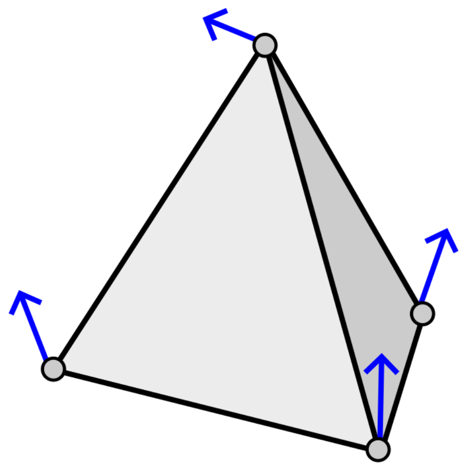
- We let the numerical solution build the space-time trajectory for us

(+) Allows to model complex behavior

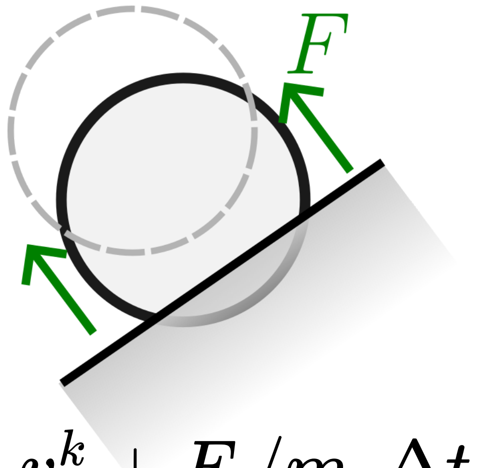
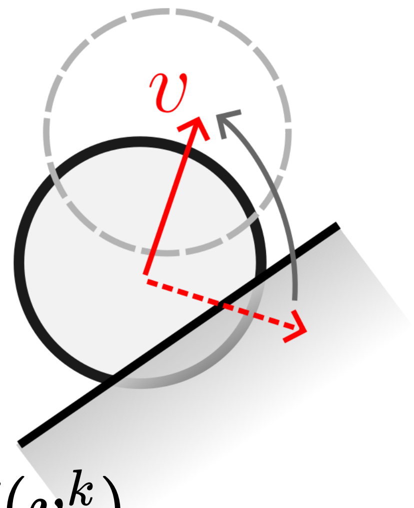
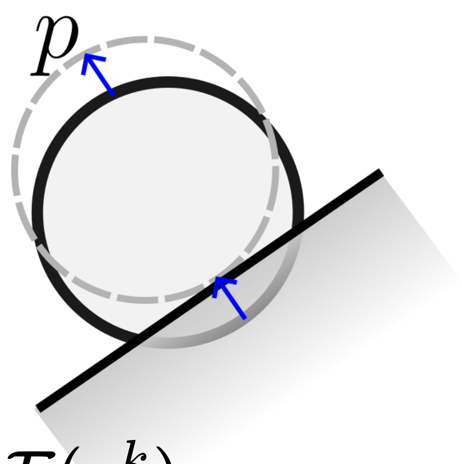
(-) Lack of control on the result

Types of Simulation Models

Deformable model

Particle	Solid	Deformable
 (p_i, v_i)	 (p_i, v_i, R_i, L_i)	 $\sigma = C\,\epsilon$

Collision/Constraint Handling

Force based	Impulse based	Position based
 $v_i^{k+1} = v_i^k + F_i/m_i\,\Delta t$	 $v_i^{k+1} \rightarrow \mathcal{F}(v^k)$	 $p_i^{k+1} \rightarrow \mathcal{F}(p^k)$

Fundamental models

1- Particles
2- Rigid bodies
3- Continuum models

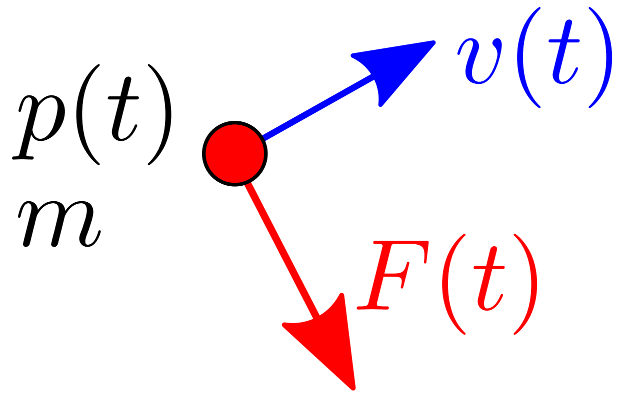
Physically-based particle system

1. Description

Particle is fully described by: Position p, Velocity v, Mass m

Fundamental quantities: position and linear momentum $P = m\,v$

Linear Momentum preserved in isolated system



2. Evolution

- Fundamental principle of dynamics

  Force applied on particle $F(p, v, t)$

$$\begin{cases} p'(t) = v(t) \\ P'(t) = m\,v'(t) = F(p, v, t) \end{cases}$$

- Conservation of energy (ex. kinetic energy ($1/2\,m\,v^2$)+potential energy = const, etc.)

- Lagragian, or Hamiltonian (reduced coordinates)

Physically-based particle system

3. Numerical Solution

ODE (Ordinary Differential Equation) formulated as an **Initial Value Problem**

$$\text{ex.} \left\{ \begin{array}{l} p'(t) = v(t) \\ mv'(t) = F(p, v, t) \end{array} \right. \quad , \quad \text{with } v(0) = v_0,\ p(0) = p_0$$

- Discretize in time $t^k = k\,h, \quad h = \Delta t =$ time step.

  $\Rightarrow$ Build a discrete numerical solution $p^k = p(t^k), v^k = v(t^k)$.

- We can consider initially the following iterative scheme

$$\left\{ \begin{array}{l} v^{k+1} = v^k + h\,F(p^k, v^k, t^k) \\ p^{k+1} = p^k + h\,v^{k+1} \end{array} \right.$$

Simple to implement, reasonably OK for simple examples (more details later).

Fundamental models

1- Particles

2- Rigid bodies

3- Continuum models

Rigid body description

- Solid defined within a domain $\Omega \subset \mathbb{R}^3$
- With a density of mass $\rho(p_i)$ at each point $p_i \in \Omega$

- Total mass of the solid m

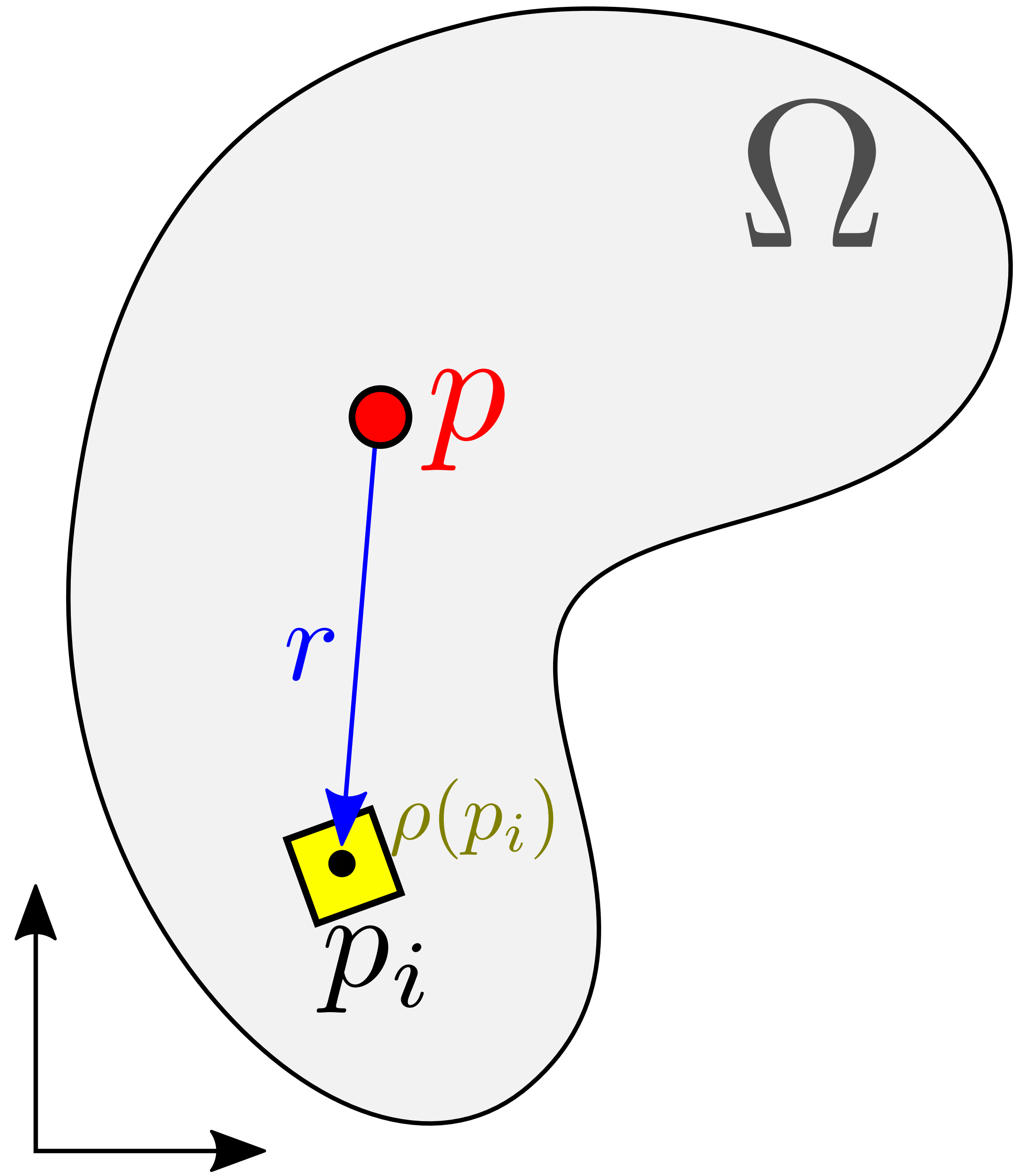
$$m = \int_{p_i \in \Omega} \rho(p_i)\, \mathrm{d}\Omega$$

- Position of the center of mass (com) p

$$p = \frac{1}{m} \int_{p_i \in \Omega} \rho(p_i)\, p_i\, \mathrm{d}\Omega$$

- Relative position of a position p_i with respect to com

$$r = p_i - p$$



Position and speed of a point on the rigid body

The center of mass has, at time t,

- a position $p(t)$
- a velocity $p'(t) = v(t)$

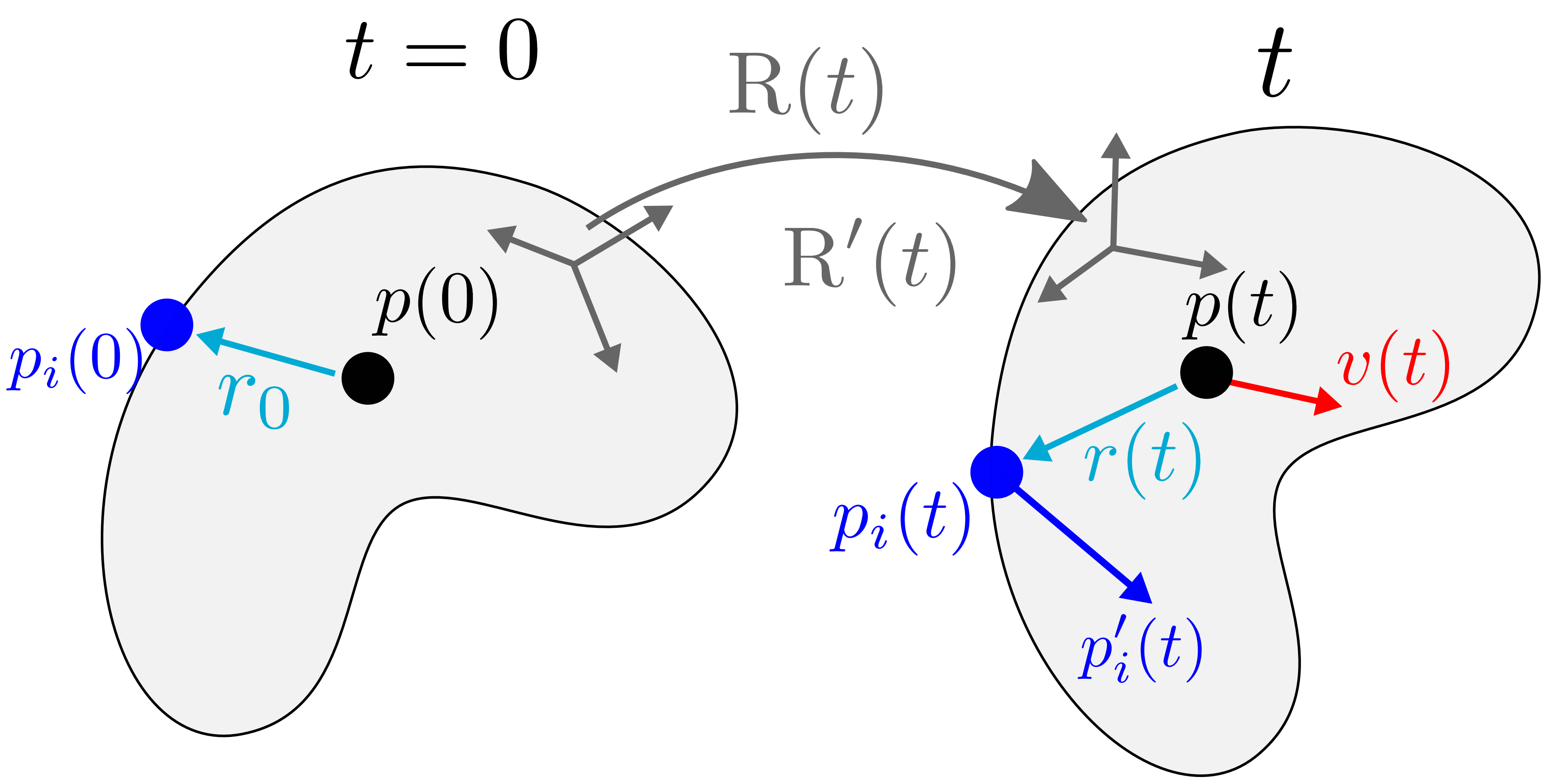
The body has an orientation $\mathrm{R}(t)$

A point of the rigid body has

- a position $\boxed{p_i(t) = p(t) + R(t)\, r_0}$

  with $r_0 = p_i(0) - p(0)$

- a speed $\boxed{p_i'(t) = v(t) + R'(t)\, r_0}$



Angular speed

Speed of p_i : $p_i'(t) = v(t) + \mathrm{R}'(t)\, r_0$

Introduce angular speed $\omega \in \mathbb{R}^3$ such that

$$\boxed{p_i'(t) = v(t) + \omega(t) \times r(t)}$$

$\omega \simeq$ *vector expressing the instantaneous rotation of* $r(t)$

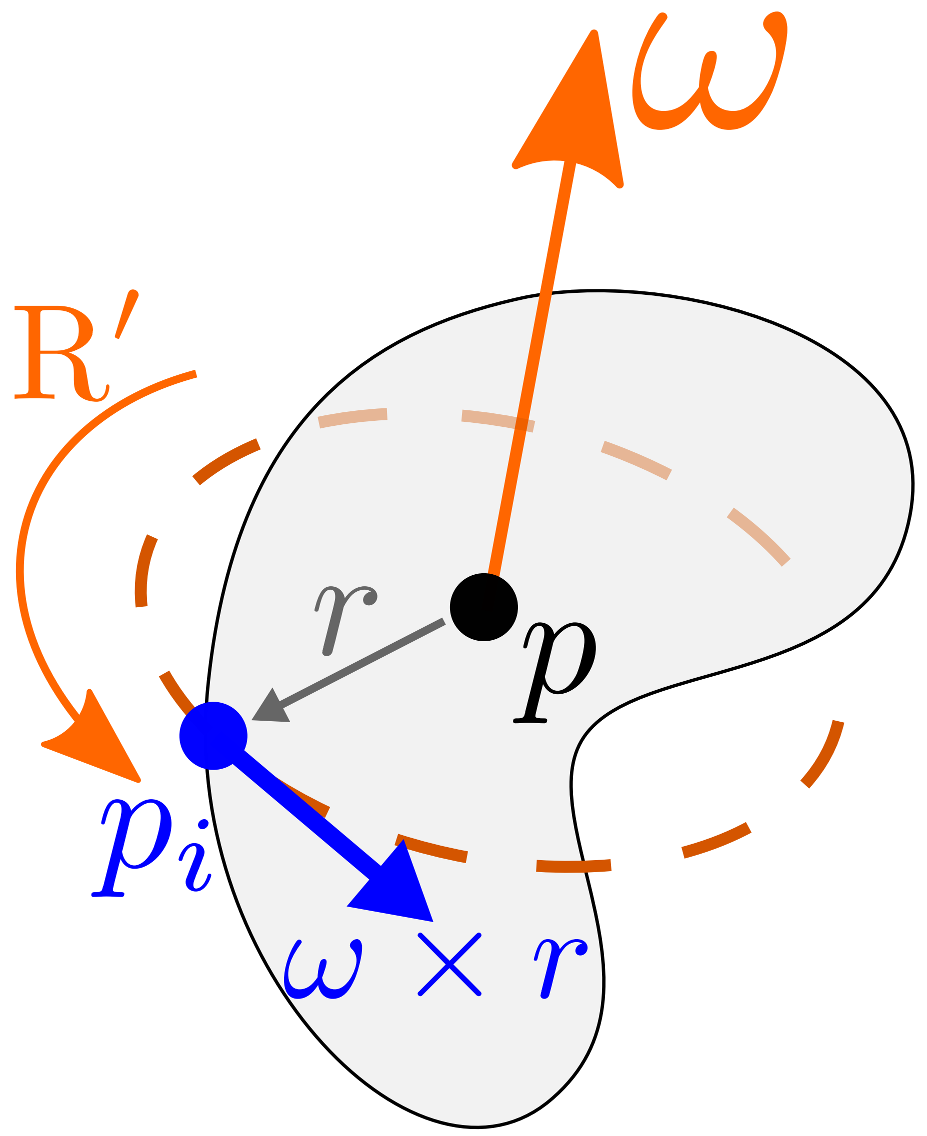
By identification $\mathrm{R}'(t)\, r_0 = \omega(t) \times r(t)$

$\Rightarrow \mathrm{R}'(t)\, r_0 = \omega(t) \times (\mathrm{R}(t)\, r_0)$

Matrix expression of $\omega = (\omega_x, \omega_y, \omega_z)$

$$\hat{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

$$\boxed{\Rightarrow \mathrm{R}'(t) = \hat{\omega}(t)\, \mathrm{R}(t)}$$



Rigid body kinematics

Similarily to particles

- Position of the com: $p(t)$
- Velocity of the com: $v(t) = p'(t)$

Linear Momentum: $\boxed{P(t) = m\, v(t)}$ $\quad \left(= \int_\Omega \rho\, v_i(t)\, \mathrm{d}\Omega) \right)$

Specific to rigid body

- Orientation of the body: $\mathrm{R}(t) \in \mathbb{R}^{3\times 3}$
- Angular velocity of the body: $\omega(t)$ such that $\hat{\omega} = \mathrm{R}'(t)\, \mathrm{R}^T(t)$

Angular Momentum: $\boxed{L(t) = \mathrm{I}(t)\, \omega(t)}$

with $\mathrm{I}(t)$: Inertia tensor

$$\mathrm{I}(t) = \mathrm{R}(t)\, \mathrm{I}_0\, \mathrm{R}^T(t)$$

$$\mathrm{I}_0 = \int_{r\in\Omega} \rho(r)\, (r^T\, r\, \mathrm{I}_d - r\, r^T)\, \mathrm{d}\Omega$$

Mass: resistance to change of speed (of the com)

Inertia: resistance to change of angular speed

Inertia tensor

Defining inertia tensor formulation from angular momentum definition

Angular momentum expressed with respect to an arbitrary point p_0: $r(p_i) = p_i - p_0$

$$L = \int_\Omega r \times (\rho\, r')\,\mathrm{d}\Omega = \int_\Omega \rho\, r \times \left(p' + \omega \times r\right)\,\mathrm{d}\Omega \quad \textit{(first part sum to 0)}$$

$$\Rightarrow L = \int_\Omega \rho\, r \times \omega \times r\,\mathrm{d}\Omega = \int_\Omega \rho\, r \times (-r \times \omega)\,\mathrm{d}\Omega$$

$$\Rightarrow L = \underbrace{\left(\int_\Omega \rho\, \hat{r}\, \hat{r}^T \mathrm{d}\Omega\right)}_{I} \omega \;=\; I\,\omega \quad \text{with } \hat{r} = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix}$$

Changing the reference frame

$$\Rightarrow L = \left(\int_\Omega \rho\, (\mathrm{R}\hat{r}_0)\, (\mathrm{R}\hat{r}_0)^T \mathrm{d}\Omega\right)\omega \;=\; \mathrm{R}\underbrace{\left(\int_\Omega \rho(r)\, \hat{r}_0\, \hat{r}_0^T \mathrm{d}\Omega\right)}_{I_0}\mathrm{R}^T\,\omega = \mathrm{R}\, I_0\, \mathrm{R}^T\,\omega$$

Inertia tensor properties

$$I = \int_{r\in\Omega} \rho(r) \begin{pmatrix} r_y^2 + r_z^2 & -r_x r_y & -r_x r_z \\ -r_x r_y & r_x^2 + r_z^2 & -r_y r_z \\ -r_x r_z & -r_y r_z & r_x^2 + r_y^2 \end{pmatrix} \mathrm{d}\Omega \quad = \quad \int_{r\in\Omega} \rho(r) \left(r^T r \; \mathrm{Id} - r\, r^T\right) \mathrm{d}\Omega$$

- I is usually expressed at the center of mass p

- I depends on the body orientation. Given a rotation R: $I = \mathrm{R}\, I_0\, \mathrm{R}^T$
  $\Rightarrow$ compute once I_0 in a rest position, then update it using R

- There exist a frame in which I is diagonal (principle axes of inertia).
  Corresponds to eigenvectors of matrix I.

Dynamics: Forces and torques on a solid

- Given a local force $f(p_i)$ acting on a position $p_i \in \Omega$
- Contribute to 2 global components applied on the body:
  - Total **net force** F applied on the shape

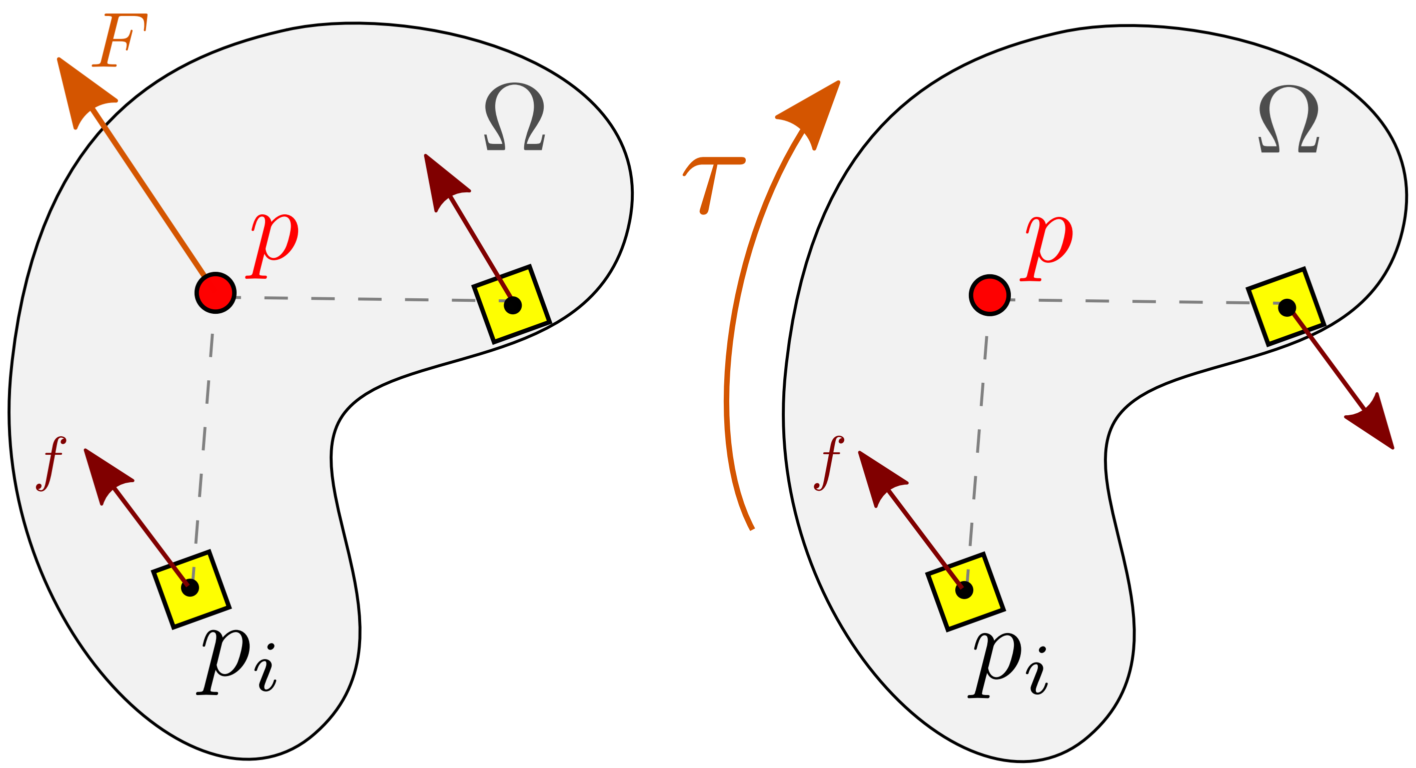
    Induce change of linear momentum (a displacement of COM)

    $$\boxed{F = \int_{p_i \in \Omega} f(p_i)\, \mathrm{d}\Omega}$$

  - **Torque** τ applied on the body

    Induce change of angular momentum (spin of the solid)

    $$\boxed{\tau = \int_{p_i \in \Omega} (p_i - p) \times f(p_i)\, \mathrm{d}\Omega}$$



Dynamics

Similarly to particles

Force F is related to the change of linear momentum $F(t) = P'(t) = (m\,v)'(t)$

Specific to rigid bodies

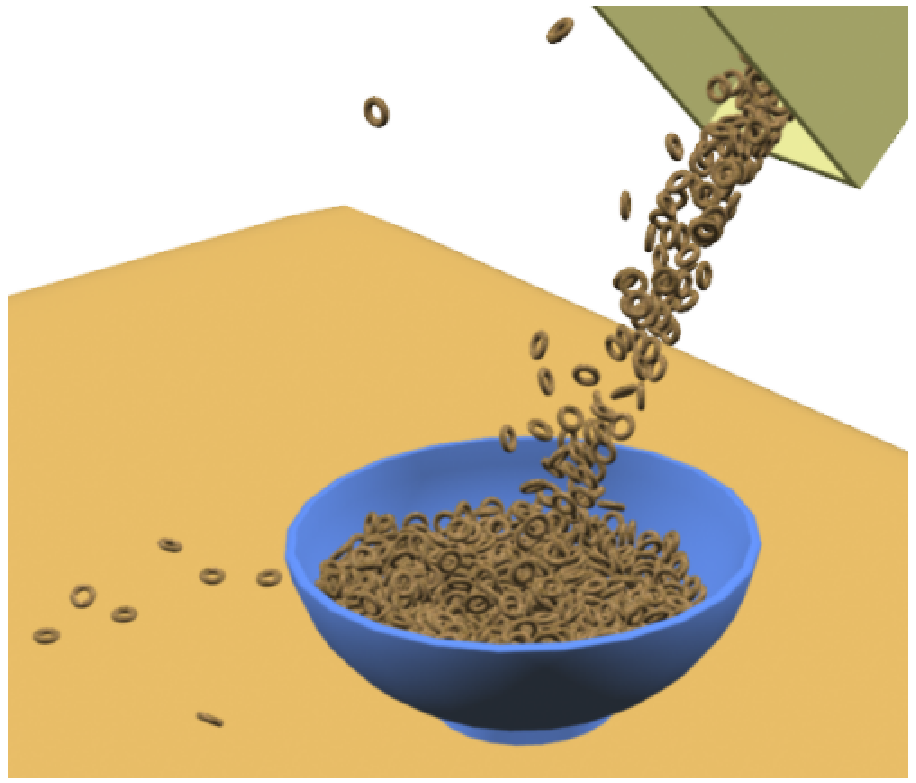
Torque τ is related to the change of angular momentum $\tau(t) = L'(t) = (I\,\omega)'(t)$

Equation of Motion

Fundamental principle of dynamics for rigid body

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p \\ P \\ \mathrm{R} \\ L \end{pmatrix} = \begin{pmatrix} v \\ F \\ \hat{\omega}\ \mathrm{R} \\ \tau \end{pmatrix}$$

Rigid solid dynamics in practice



1. Initial condition

- $p(t_0), v(t_0), \mathrm{R}(t_0), \omega(t_0)$ given as initial condition
- Precompute $I_0 = I(t_0)$
- Compute $L(t_0) = I_0\ \omega(t_0)$

2. Temporal Evolution

Iterate over time t_k

- Compute total force $F(t_k)$ and torque $\tau(t_k)$
- Compute $I(t_k) = \mathrm{R}(t_k)\, I_0\, \mathrm{R}^T(t_k)$
- Compute $\omega(t_k) = I(t_k)^{-1}\, L(t_k)$

- Numerical integration updating state vector
  $\rightarrow (p(t_{k+1}), P(t_{k+1}), \mathrm{R}(t_{k+1}), L(t_{k+1}))$

- *Handle collision*

$$\begin{pmatrix} p'(t) \\ P'(t) \\ \mathrm{R}'(t) \\ L'(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t) \\ \hat{\omega}(t)\ \mathrm{R}(t) \\ \tau(t) \end{pmatrix}$$

$$I(t) = \mathrm{R}(t)\, I_0\, \mathrm{R}^T(t)$$
$$L(t) = I(t)\, \omega(t)$$
$$P(t) = m\, v(t)$$

Side note: Use of quaternion

Relation $\mathrm{R}'(t) = \hat{\omega}(t)\,\mathrm{R}(t)$ may lead to numerical drift from rotation matrix

ex. $\mathrm{R}^{k+1} = (\mathrm{Id} + h\,\hat{\omega}^k)\,\mathrm{R}^k$ (explicit scheme)

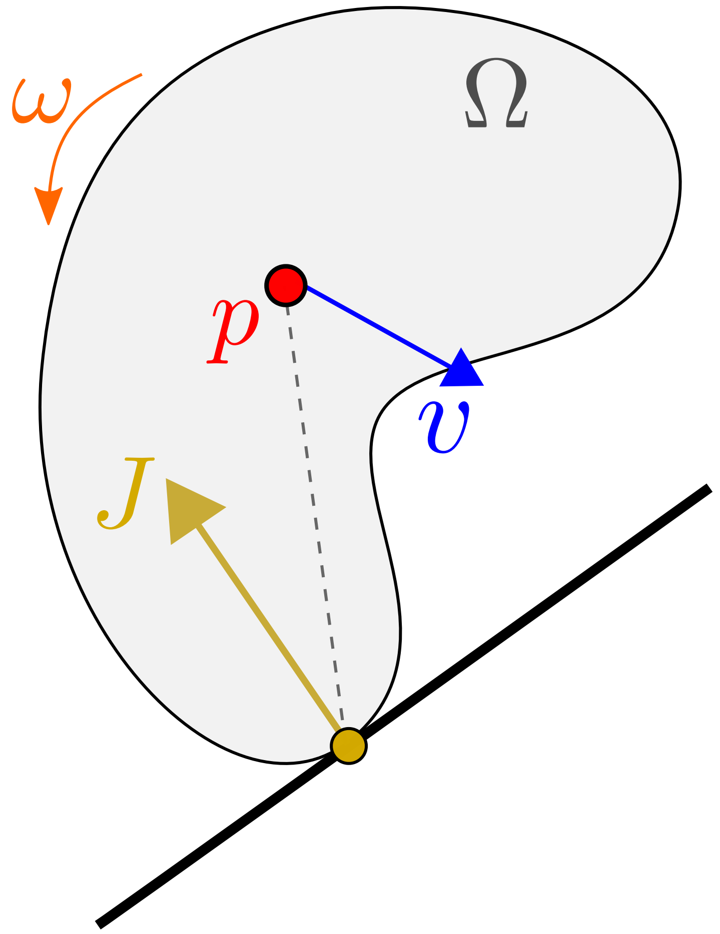
Using quaternion leads to more robust behavior

- Quaternion expression: $q'(t) = \frac{1}{2}\,q_\omega(t)\,q(t)$, with $q_\omega(t) = (\omega(t)\,,\quad 0))$

- Quaternion is forced to keep a unit norm

$$\text{ex.} \left\{ \begin{array}{l} q^{k+1} = q^k + \frac{1}{2}\,q_\omega^k\,q^k \\ q^{k+1} = q^{k+1}/\|q^{k+1}\| \end{array} \right.$$

Collisions in rigid bodies



Collisions at position p change both linear and angular velocity

Use of impulse (sudden change of velocity) J.

$J = \int_{t_0}^{t_0+\Delta t} F(t)\mathrm{d}t$

Impulse split into

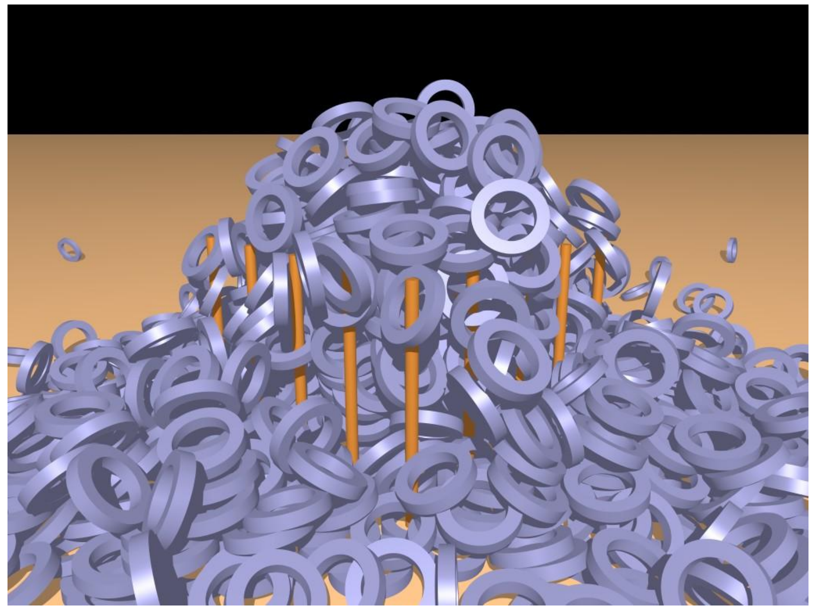
- *Force impulse* $\Delta P = m\,\Delta v = J$
- *Torque impulse* $\Delta L = I\,\Delta\omega = (p - p_{com}) \times J$

Elastic collision between two solids at position p_i:

$J = j\,n$, n: normal of the separating planes

$j = (v_1(p_i) - v_2(p_i)) \cdot n / K$

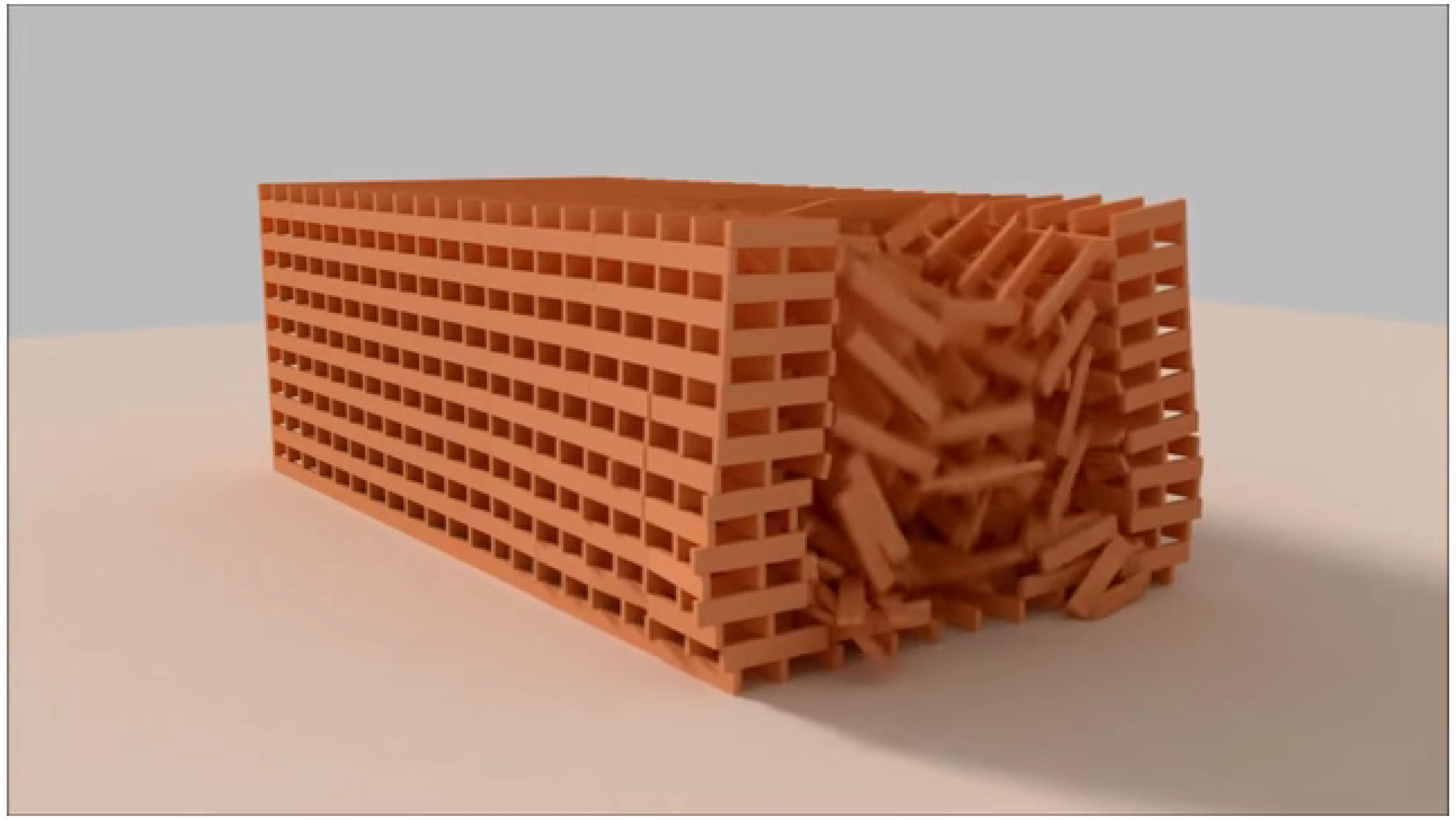
$K = 1/m_1 + 1/m_2 + n \cdot \left(I_1^{-1}(r_1 \times n) \times r_1 + I_2^{-1}(r_2 \times n) \times r_2\right)$



More details [D. Barff. Physically Based Modeling. SIGGRAPH Course Notes 1999]

Rigid bodies usage

- Standard usage for rigid bodies motions
  - Limited to non-deformable shapes
- Common in VFX (explosions), and *simulation games* (cars, airplanes, etc).
- Standard library: Bullet physics (ex. used in Blender).



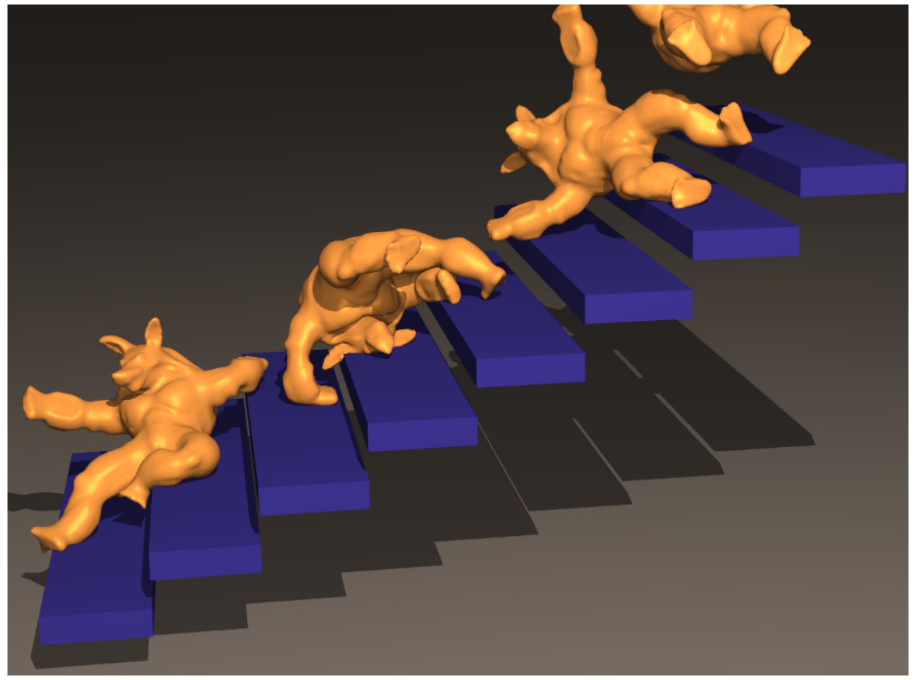
Fundamental models

1- Particles

2- Rigid bodies

3- Continuum material

Deformation of a continuous shape



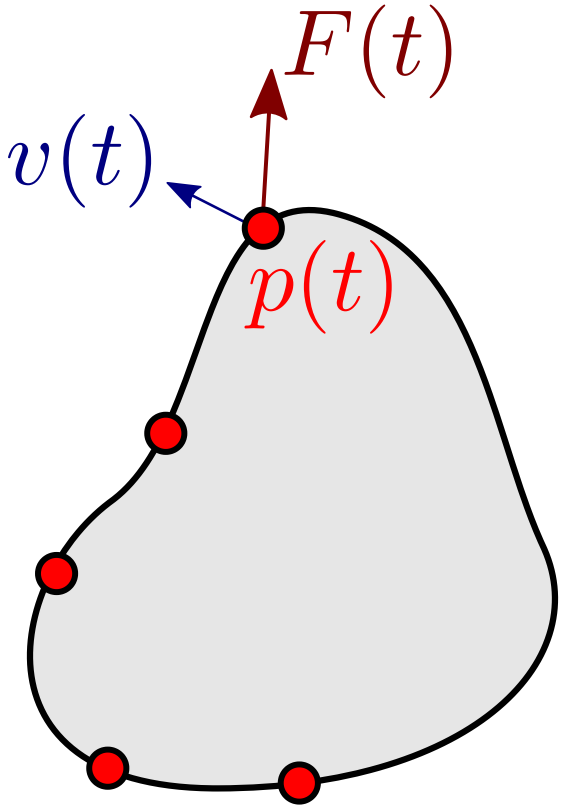
Every part of the shape can be deformed

ex. Describing elastic shapes, visco-elastic shapes, fluids, etc.

Two ways to describe the deforming object

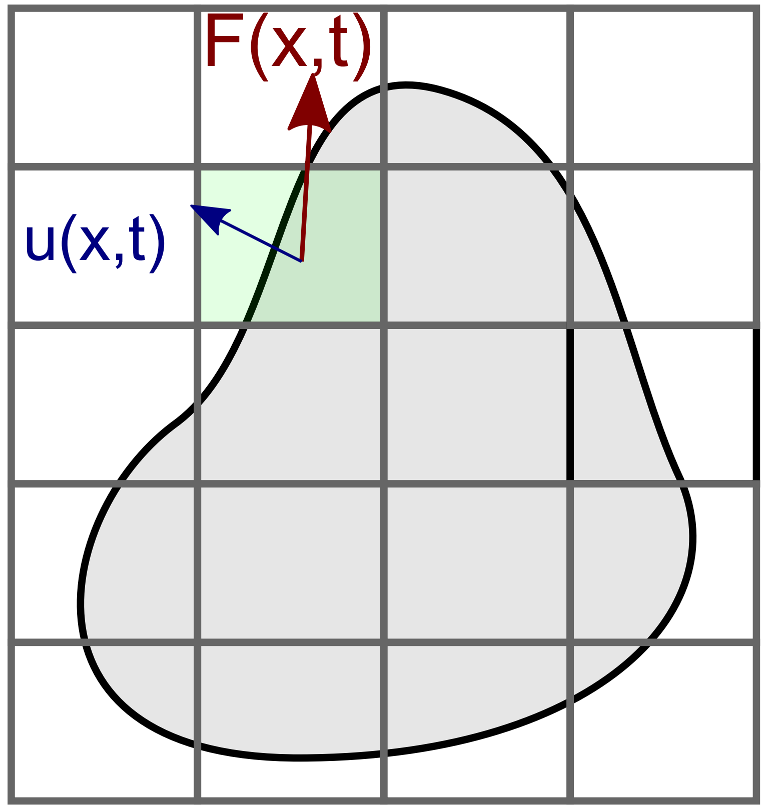
1. Lagrange representation

Positions follow the object deformation



2. Euler representation

Positions are fixed in 3D space

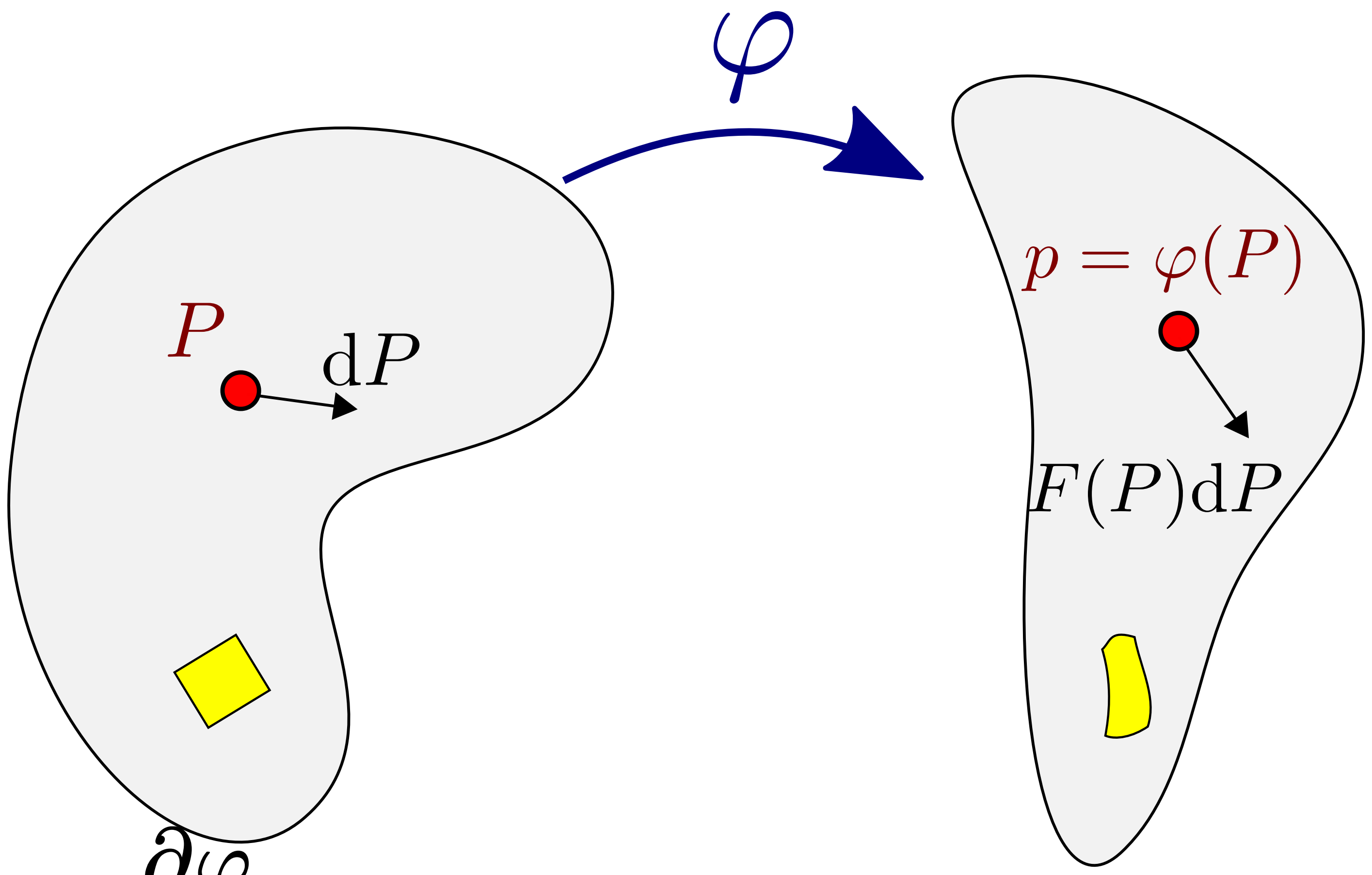


Deformation the Lagrangian description

Deformation map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $p = \varphi(P)$

P position in the reference undeformed shape

p position in the deformed configuration.



Deformation Gradient F

- $F(P) = \dfrac{\partial \varphi}{\partial P}(P) = \dfrac{\partial p}{\partial P} \in \mathbb{R}^{3\times 3}$
- Characterizes the local deformation associated to φ

Position $P + \mathrm{d}P$ is mapped into $\varphi(P + \mathrm{d}P) \simeq p + \dfrac{\partial \varphi}{\partial P}\,\mathrm{d}P$

- $F(P) = \begin{pmatrix} \frac{\partial \varphi_x}{\partial X} & \frac{\partial \varphi_x}{\partial Y} & \frac{\partial \varphi_x}{\partial Z} \\ \frac{\partial \varphi_y}{\partial X} & \frac{\partial \varphi_y}{\partial Y} & \frac{\partial \varphi_y}{\partial Z} \\ \frac{\partial \varphi_z}{\partial X} & \frac{\partial \varphi_z}{\partial Y} & \frac{\partial \varphi_z}{\partial Z} \end{pmatrix}$

Strain

Deformation gradient F describe both

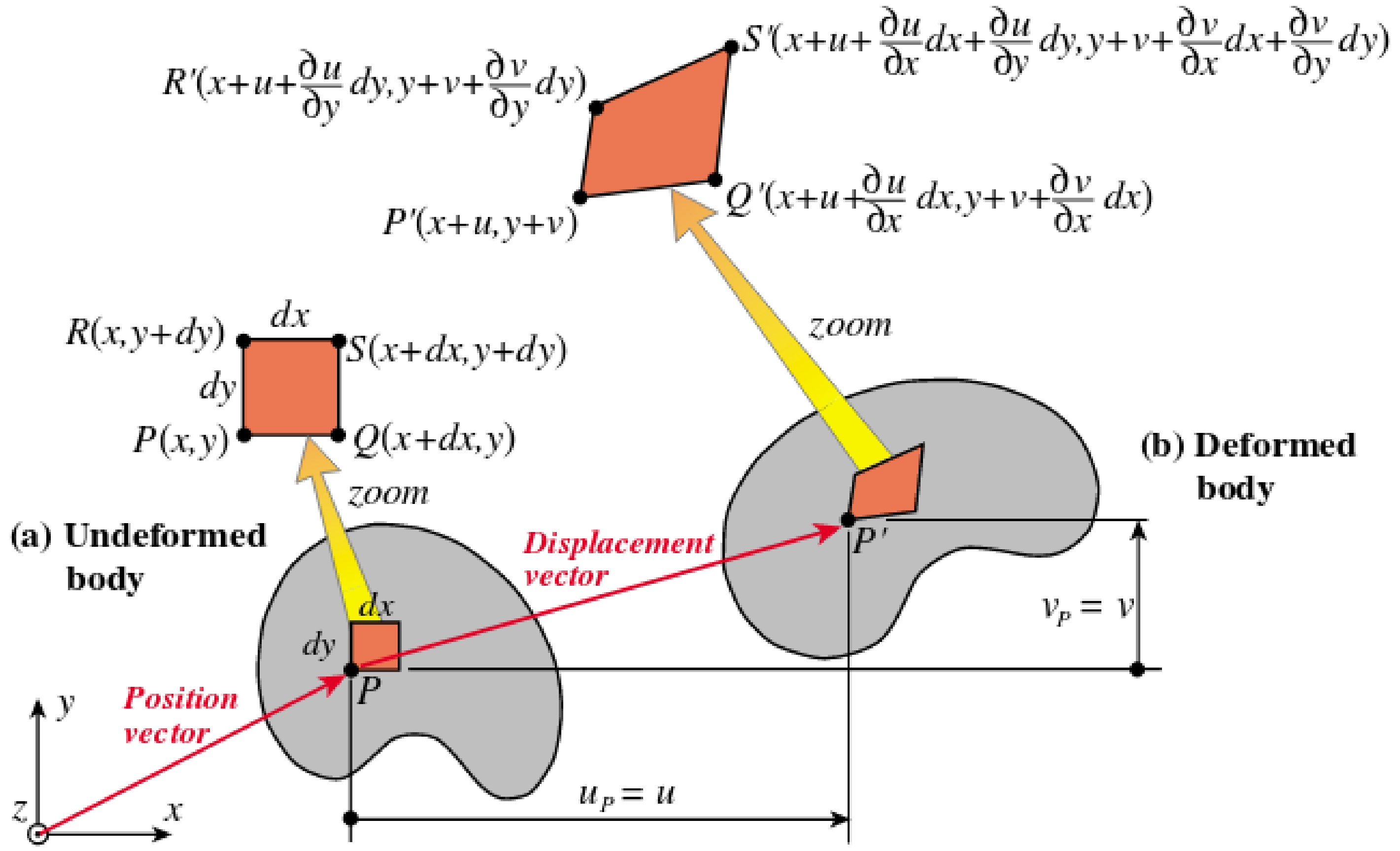
- Rigid transformation (rotation) - not related to material effort
- Any other deformation inducing local length change - related to material effort

Strain ϵ is a measure of deformation ignoring rigid transformation.

Several possible measure of strain

- Green strain tensor $\epsilon = \frac{1}{2}\,(F\,F^T - \mathrm{Id})$
  - *(+) If* φ *is a rotation* $F = R \Rightarrow \epsilon = 0$
  - *(-) Non linear in* p

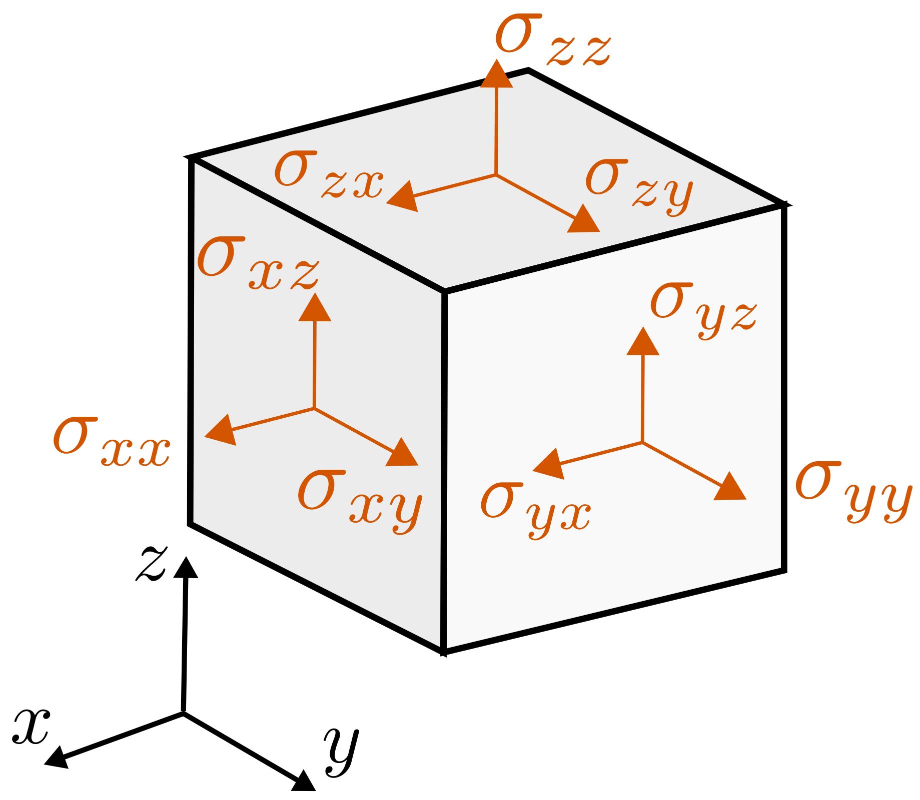
- Linearized Cauchy strain $\epsilon = \frac{1}{2}(F^T + F) - \mathrm{Id}$
  - *Used for small deformations*



Stress

Stress $\sigma \in \mathbb{R}^{3\times 3}$ describes internal forces (per area unit) induced by the local deformation (strain) in any direction

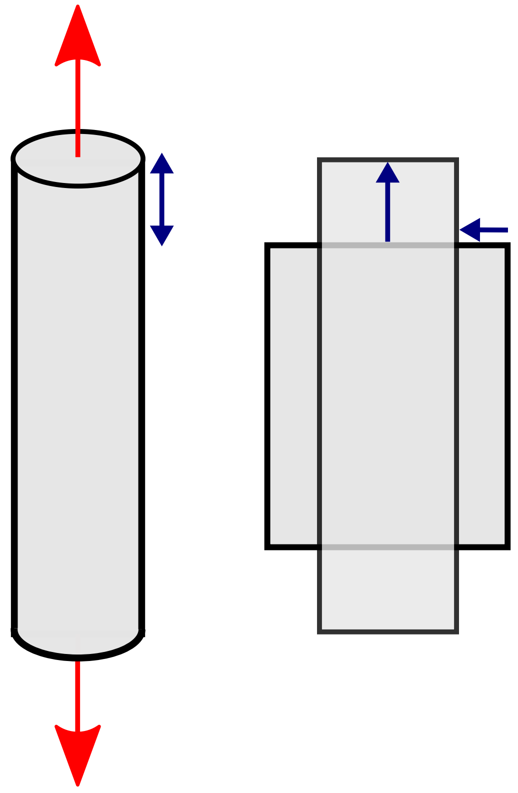
Constitutive Relation: Relation between stress and strain, characterize a type of material.



For linear constitutive relation:

$$\sigma_{ij} = \sum_{k,l} C_{ijkl}\, \epsilon_{kl} \quad , \; C\text{: stiffness tensor (81 coefficients)}$$

Strain energy/elastic potential energy: $U = \frac{1}{2}\sum_{i,j,k,l} \sigma_{ij}(\epsilon)\epsilon_{kl} = \frac{1}{2}\sum_{i,j,k,l} C_{ijkl}\,\epsilon_{ij}\,\epsilon_{kl}$



For homogeneous isotropic elastic material, constitutive relation simplifies to

$\sigma = 2\mu\,\epsilon + \lambda \mathrm{tr}(\epsilon)\,\mathrm{Id}$, (μ, λ): Lamé parameters

Related to common mechanical modulus : Young' modulus Y and Poisson's ratio ν

$\mu = \frac{Y}{2(1+\nu)}, \lambda = \frac{Y\nu}{(1+\nu)(1-2\nu)}$

Evolution equation

Fundamental principle of dynamics in the entire volume Ω

Change of momentum = External forces (in volume) + Traction (stress applied on exterior surface normals)

$$\Rightarrow \underbrace{\int_\Omega \rho\, p''(t)\,\mathrm{d}\Omega}_{\textit{Change of momentum}} = \underbrace{\int_\Omega F(t)\,\mathrm{d}\Omega}_{\textit{External forces}} + \underbrace{\int_{\partial\Omega} \sigma\, n\,\mathrm{d}S}_{\textit{Traction force on the boundary}}$$

Using divergence theorem $\int_{\partial\Omega} \sigma\, n\,\mathrm{d}S = \int_\Omega div(\sigma)\,\mathrm{d}\Omega$

Equation in volume satisfied at each position $p \in \Omega$

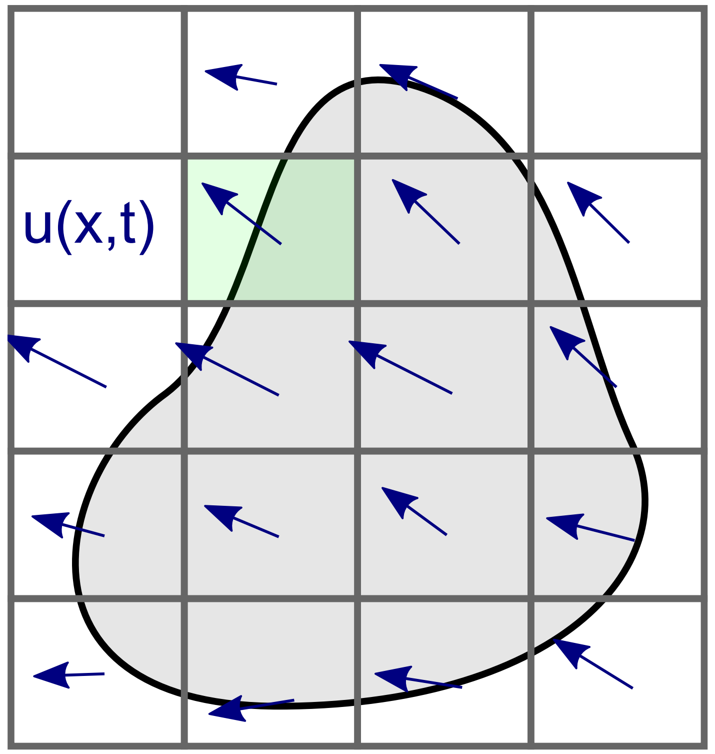
$$\boxed{\rho\, p''(t) = F(t) + div(\sigma(t))} \quad \sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \quad div(\sigma) = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

Euler formulation

In Euler formulation quantities are expressed at fixed position in 3D space.

Deformation described by velocity $u(p,t)$ at a given 3D fixed point $p = (x,y,z)$ at time t.

- Do not require anymore a reference shape
- Usefull for heavily deforming shapes (ex. fluids, gaz).



- Change of speed during $\mathrm{d}t$

$$\frac{\mathrm{d}u}{\mathrm{d}t}(p,t) = \frac{\partial u}{\partial t} + \sum_i \frac{\partial u}{\partial p_i} \underbrace{\frac{\mathrm{d}p_i}{\mathrm{d}t}}_{u_i} = \frac{\partial u}{\partial t} + (u \cdot \nabla) u$$

Called *material derivative*.

- Similarily to Lagrangian derivation:
  - Strain-rate tensor ϵ (rate of change of deformation in a neighborhood of a point) expressed with respect to u: $\epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$
  - Stress-rate tensor σ (rate of change of direction force per area in a neighborhood of a point).

Equation of motion for a fluid

- Fundamental principle of dynamics on linear momentum

$$\rho \frac{\mathrm{d}u}{\mathrm{d}t} = F + \mathrm{div}(\sigma)$$

$$\Rightarrow \rho \, \frac{\partial u}{\partial t} = F + \mathrm{div}(\sigma) - \rho \, (u \cdot \nabla) u.$$

The term $(u \cdot \nabla) u$ is called *advection*.

- External force: weight $F = \rho \, g$
- Stress decomposed into

$$\sigma = \sigma_{viscous} + \sigma_{pressure}$$

$\sigma_{pressure} = -p \, \mathrm{Id}$ *(pressure acts along normal of surface elements)*

$$\rho \, \frac{\partial u}{\partial t} = \rho \, g - \rho \, u \cdot \nabla u + \mathrm{div} \left(\sigma_{viscous} - p \, \mathrm{Id} \right)$$

$$\Rightarrow \rho \, \frac{\partial u}{\partial t} = \rho \, g - \rho \, u \cdot \nabla u - \nabla p + \mathrm{div} \left(\sigma_{viscous} \right)$$

Navier-Stokes equation

- Isotropic Newtonian fluid $\Rightarrow$ Linear (scalar) relation between strain-rate ϵ and stress-rate $\sigma_{viscous}$
  - $\sigma_{viscous} = 2\mu\,\epsilon = \mu\left(\nabla u + \nabla u^T\right)$, μ constant viscosity parameter
- Incompressible fluid $\Rightarrow \mathrm{div}(u) = 0$

Equation of motion

$$\Rightarrow \rho\,\frac{\partial u}{\partial t} = \rho\,g - \rho\,u\cdot\nabla u - \nabla p + \mathrm{div}\left(\mu\left(\nabla u + \nabla u^T\right)\right)$$

- Noting that $\mathrm{div}(\nabla u^T) = \nabla\,\mathrm{div}(u) = 0$
- And $\mathrm{div}(\nabla u) = \triangle u$
- Set $\nu = \mu/\rho$

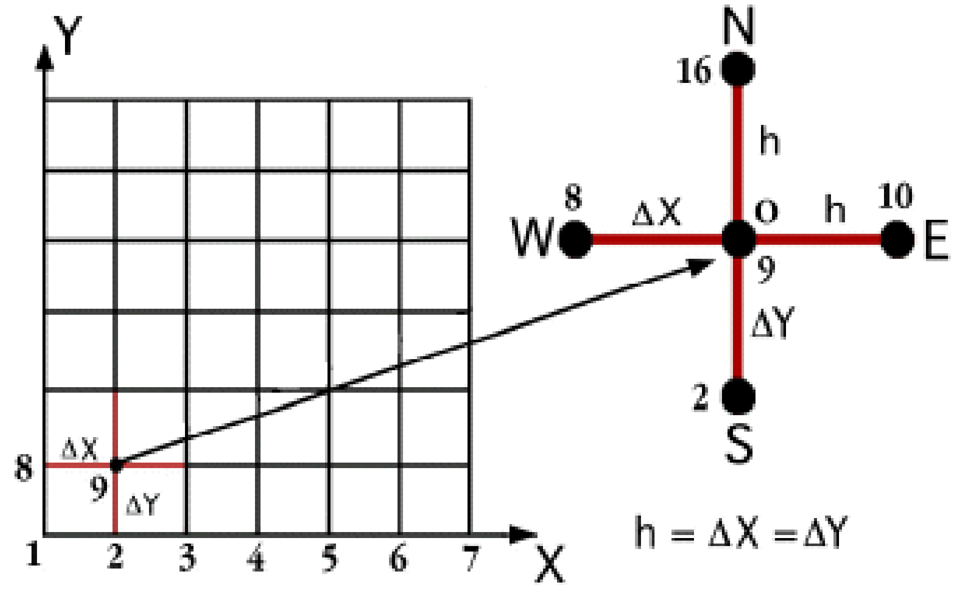
$$\boxed{\Rightarrow \frac{\partial u}{\partial t} = g - (u\cdot\nabla)u - \frac{1}{\rho}\nabla p + \nu\triangle u}$$

Navier-Stokes equation for incompressible Newtonian fluid.

Numerical solutions

Lagrangian and Euler description leads to PDE (Partial Differential Equations)

In general: no explicit solutions $\Rightarrow$ approximate numerical solution



Finite Differences (FD)

- Discretize in space on a grid Δx and time Δt
- Use numerical approximation of derivatives using masks in space and time
  - (+) Very general, simple to setup
  - (+) Works well with rectangular grid (ex. Euler description)
  - (-) Difficult to handle shape boundaries
  - (-) Instabilities

Finite elements method (FEM)

- Discretize the shape into simple elements. Build continuous function on each element.
  In CG: Elements are tetrahedron (in volume). Continuous function are barycentric coordinates (linear interpolation functions).
- Integrate PDE over each element (weak formulation), leads to a linear system
  - (+) Handle boundaries (ex. Deforming solid)
  - (+) Guarantee on accuracy
  - (-) Complex to set up, and computationally heavy
  - (-) Requires good quality meshing