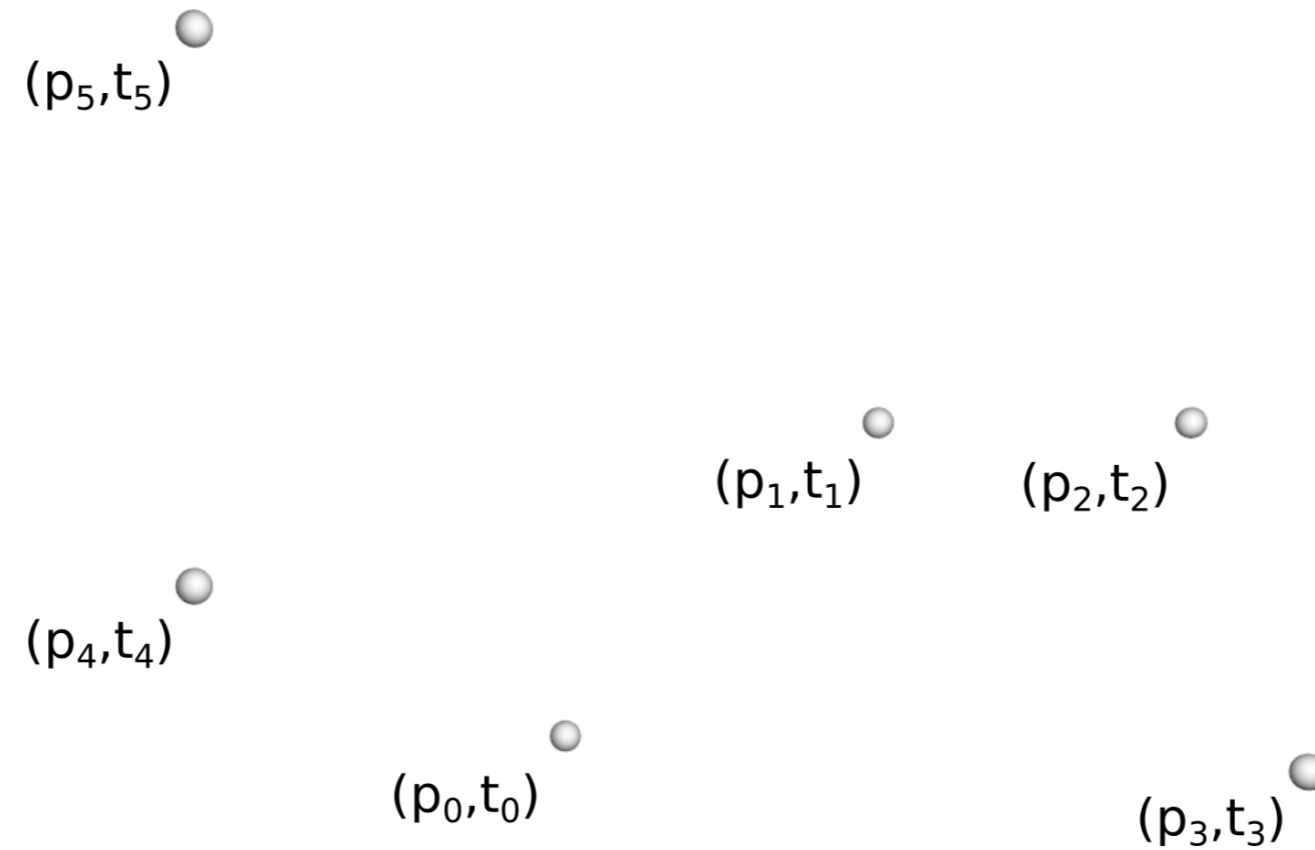


# Interpolating positions

*Spline trajectory*

# Objective

- Given a set of key-positions (positions+time) we want to find an interpolating space-time curve



- **Input**  $(p_i, t_i), i \in [0, N - 1]$
- **Output**
  - Space time curve  $p(t), t \in [t_0, t_{N-1}]$
  - Space time curve  $p(t_i) = p_i$

# Linear Interpolation

- Simplest solution: linear interpolation between each sample pairs  $(p_i, p_{i+1})$

$$- \forall t \in [t_i, t_{i+1}], \begin{cases} p(t) = (1 - \alpha(t)) p_i + \alpha(t) p_{i+1} \\ \alpha(t) = \frac{t - t_i}{t_{i+1} - t_i} \end{cases}$$

## Pros

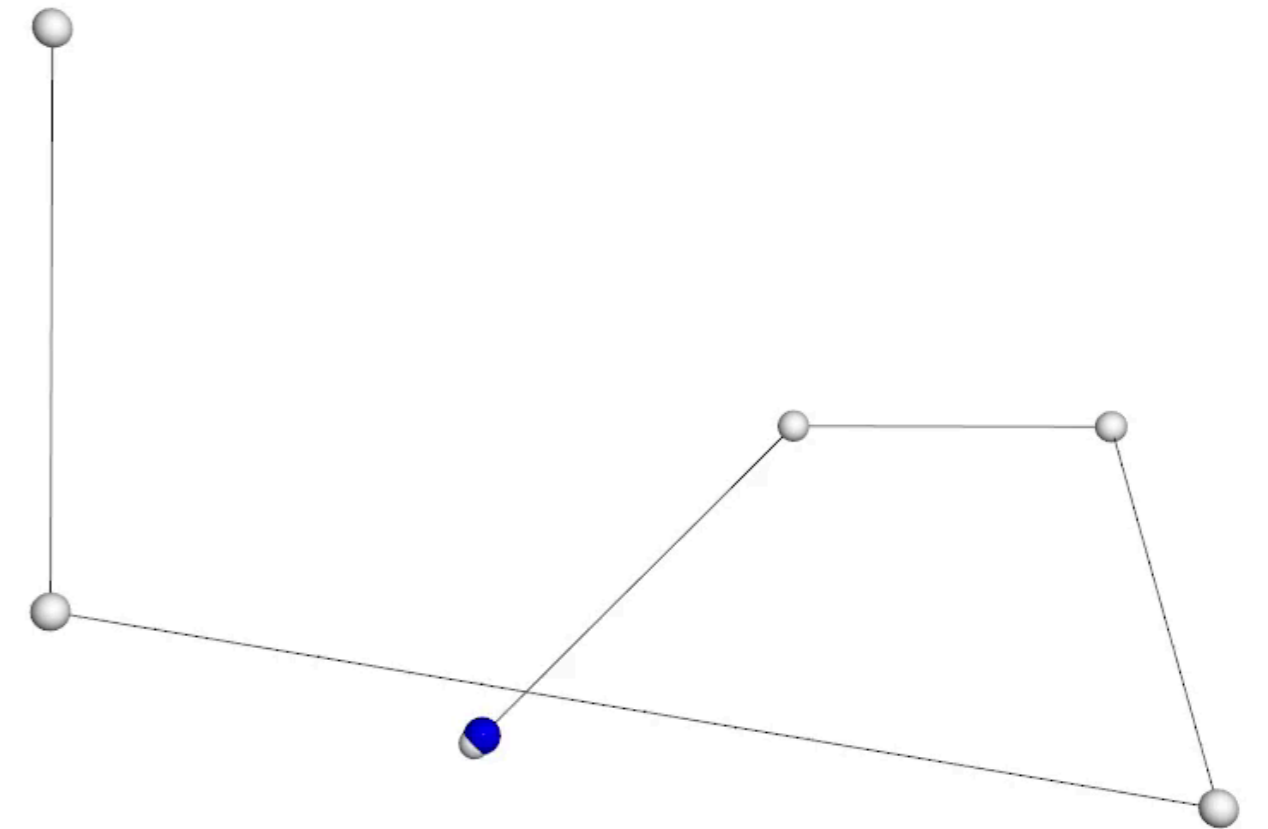
- Simple
- Constant speed between keyframes

*Easy to adjust*

## Cons

- Non smooth trajectory
- Generates straight segments

*Artists prefer non straight trajectories for "real" looking motion*



# Smooth curve

**Objective:** Generate a **smooth** interpolating space-time curve

*Classical Idea* Use polynomials curves

$$\left\{ \begin{array}{l} p(t) = \sum_{i=0}^{N-1} \alpha_i(t) p_i \\ \alpha_i(t) = \sum_{j=0}^d c_i^j t^j \end{array} \right.$$

$(\alpha_i)$  polynomial basis function of degree  $d$

*Which polynomials/degree choose ?*

# Lagrange polynomial interpolation

**Naive idea: Interpolate all points at once**

$$\forall t \in [t_0, t_{N-1}], \quad p(t) = \sum_{i=0}^{N-1} \alpha_i(t) p_i \quad \forall i \in [0, N-1] \quad p(t_i) = p_i$$

Degree of polynomial :  $N - 1$

- Known solution: **Lagrange polynomial**

$$p(t) = \sum_{i=0}^{N-1} \alpha_i(t) p_i \quad \alpha_i(t) = \prod_{k=0, k \neq i}^{N-1} \frac{t - t_k}{t_i - t_k}$$

- Explanation: By construction  $\alpha_i(t_i) = 1$  and  $\alpha_i(t_k) = 0$

(+) Interpolate all points

(-) Large oscillations between samples for large degree.

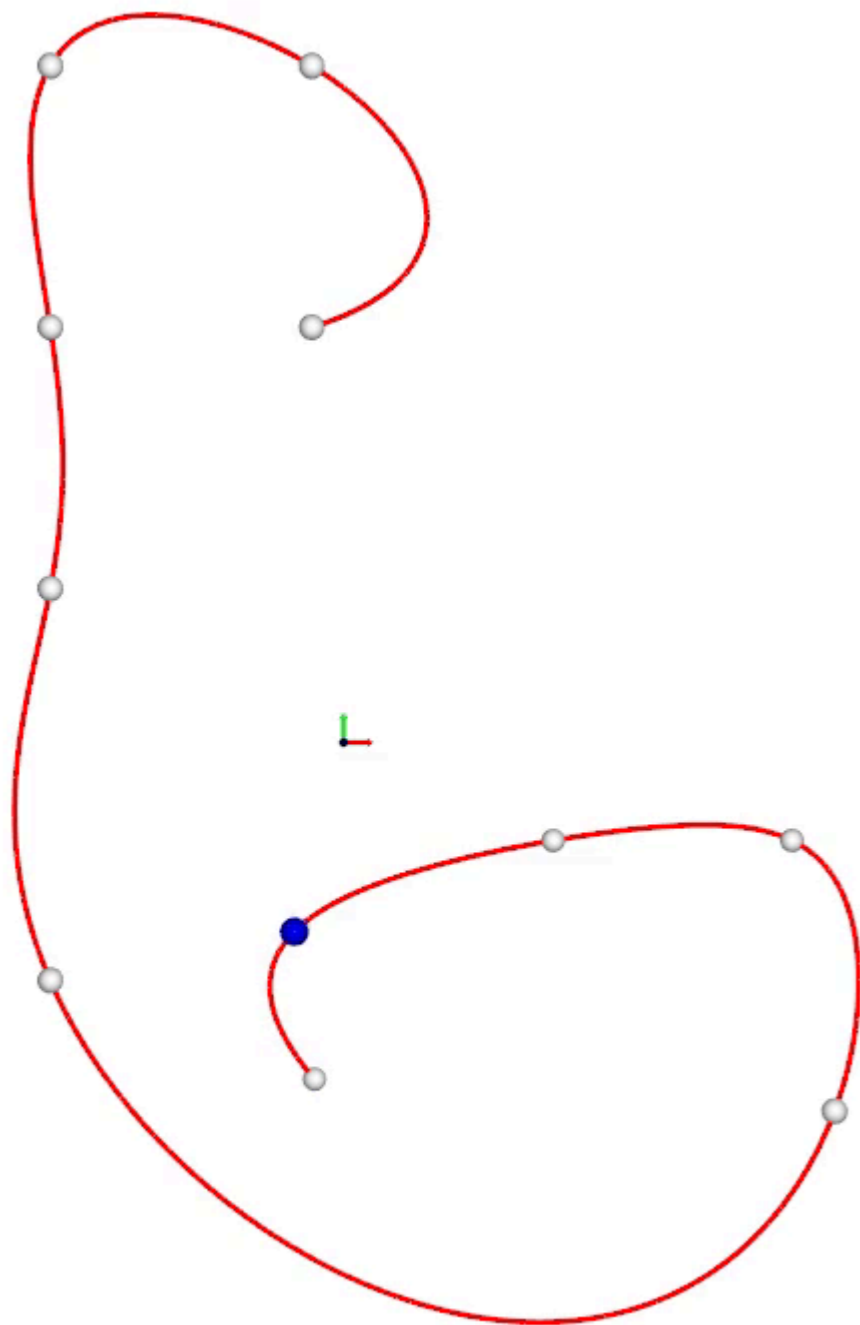
(-) Non local influence

=> Not used in practice

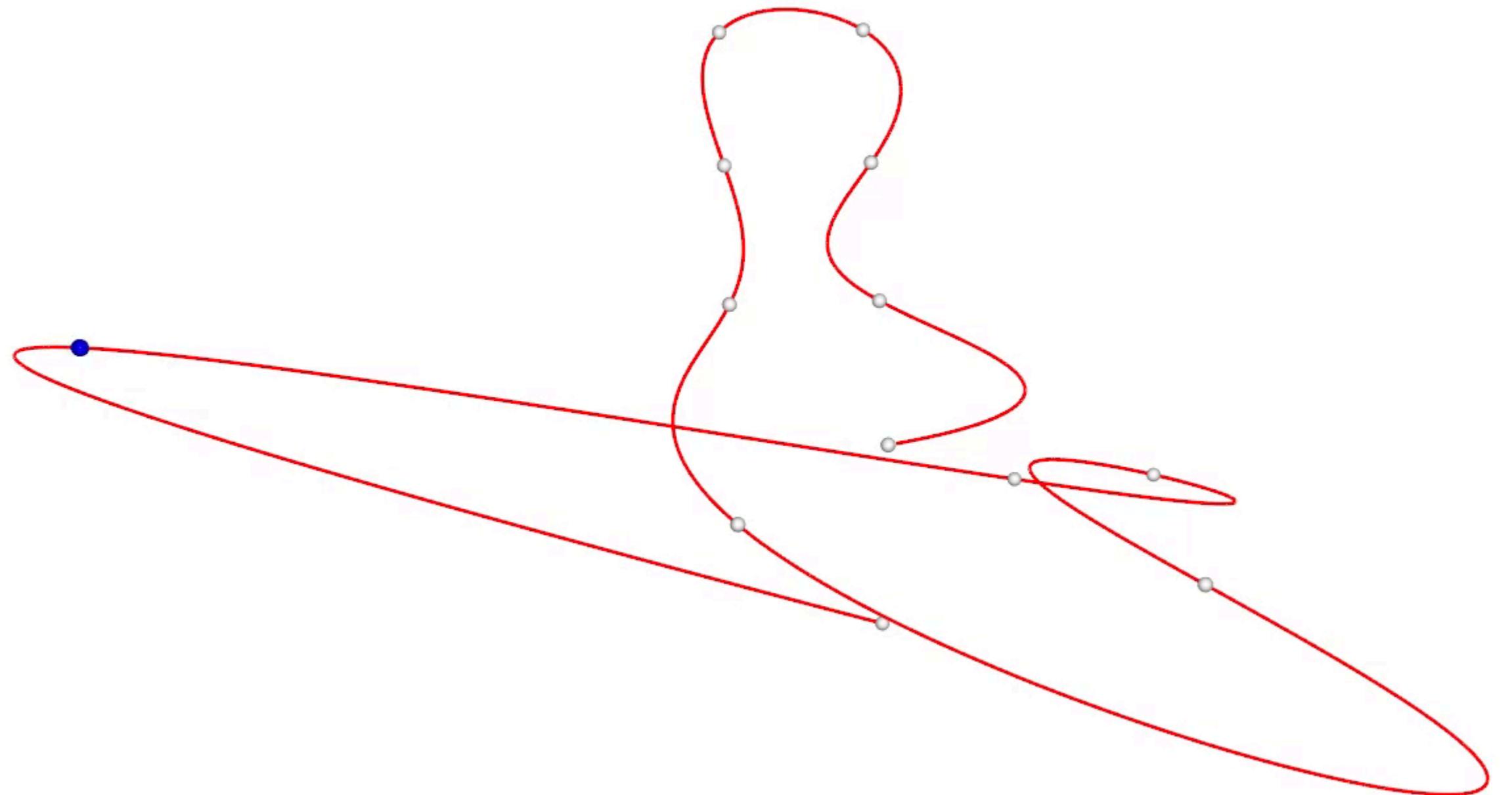
# Lagrange polynomial interpolation : Comparison

*Ex. Global effect on curve when two samples are added.*

10 samples



12 samples



# Spline

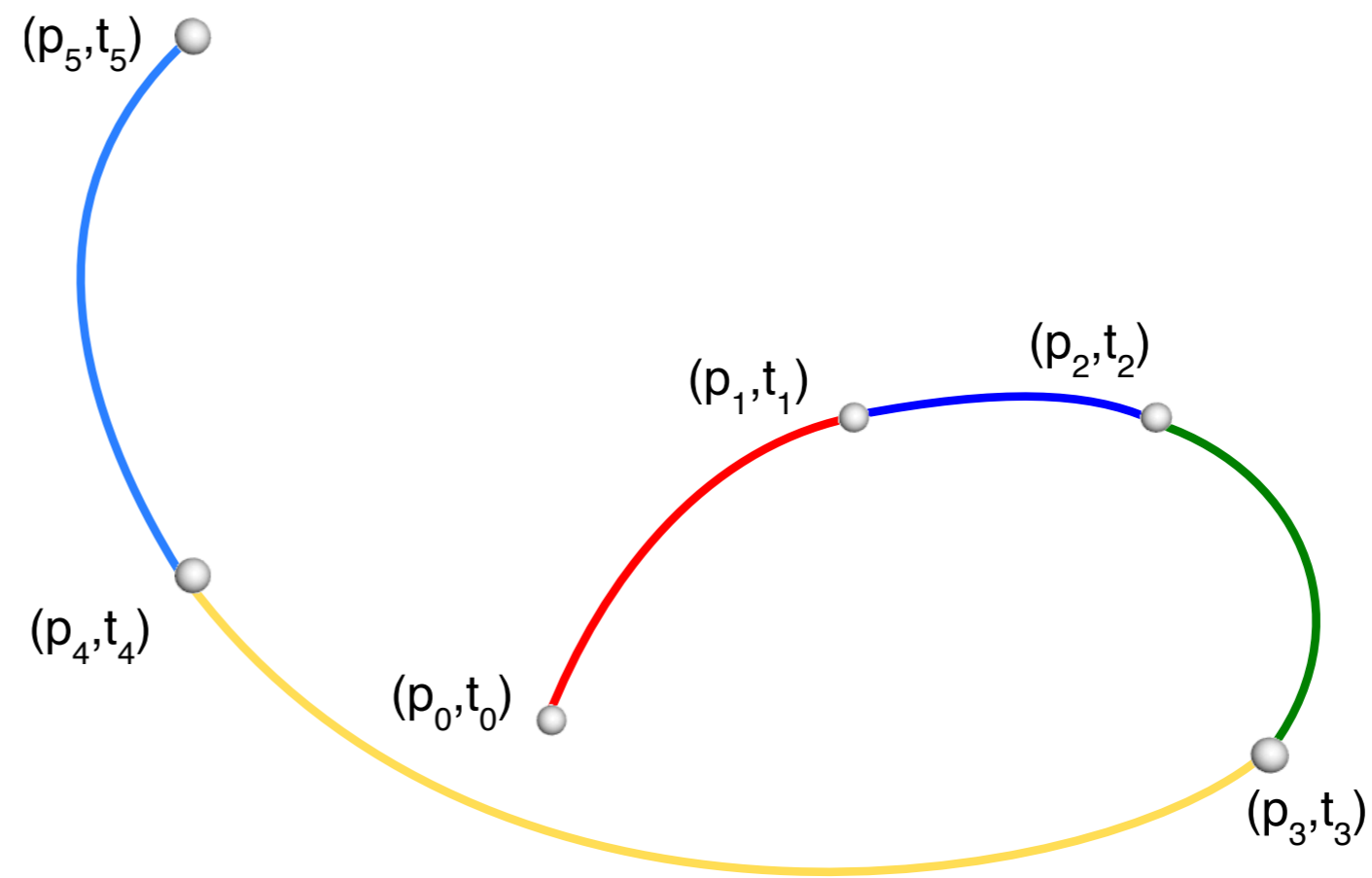
## Idea

- Define on each part a polynomial
- Smooth junctions between them.

## How to choose the polynomial

- Sufficiently high degree to be smooth
- Sufficiently low degree to avoid oscillations

=> In Graphics cubic polynomials are often used  
*Allows up to  $C^2$  junctions*



*Each color is another polynomial*

# Hermite interpolation

Hermite interpolation : cubic curve interpolating points and derivatives at extremities

Consider the following constraints

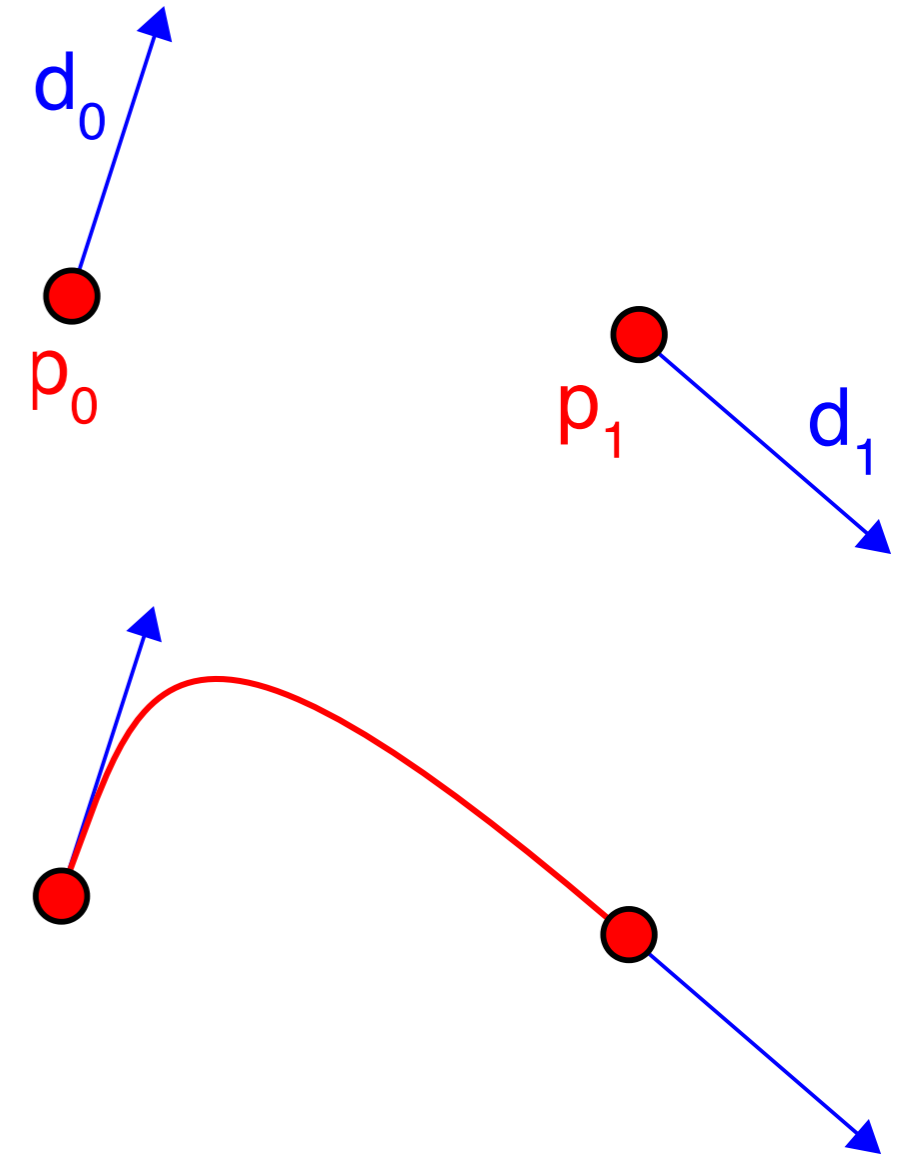
$$\begin{cases} p(s) = c_3 s^3 + c_2 s^2 + c_1 s + c_0 \\ p(0) = p_0, p(1) = p_1, p'(0) = d_0, p'(1) = d_1 \end{cases}$$

⇒ System of equations

$$\begin{cases} c_0 = p_0 \\ c_3 + c_2 + c_1 + c_0 = p_1 \\ c_1 = d_0 \\ 3c_3 + 2c_2 + c_1 = d_1 \end{cases} \Rightarrow \begin{cases} c_0 = p_0 \\ c_1 = d_0 \\ c_2 = -3p_0 + 3p_1 - 2d_0 - d_1 \\ c_3 = 2p_0 - 2p_1 + d_0 + d_1 \end{cases}$$

$$\forall s \in [0, 1], p(s) = (2s^3 - 3s^2 + 1) p_0 + (s^3 - 2s^2 + s) d_0 + (-2s^3 + 3s^2) p_1 + (s^3 - s^2) d_1$$

For arbitrary  $t \in [t_i, t_{i+1}]$ , we set  $s = \frac{t - t_i}{t_{i+1} - t_i}$





# Wrap-up Algorithm

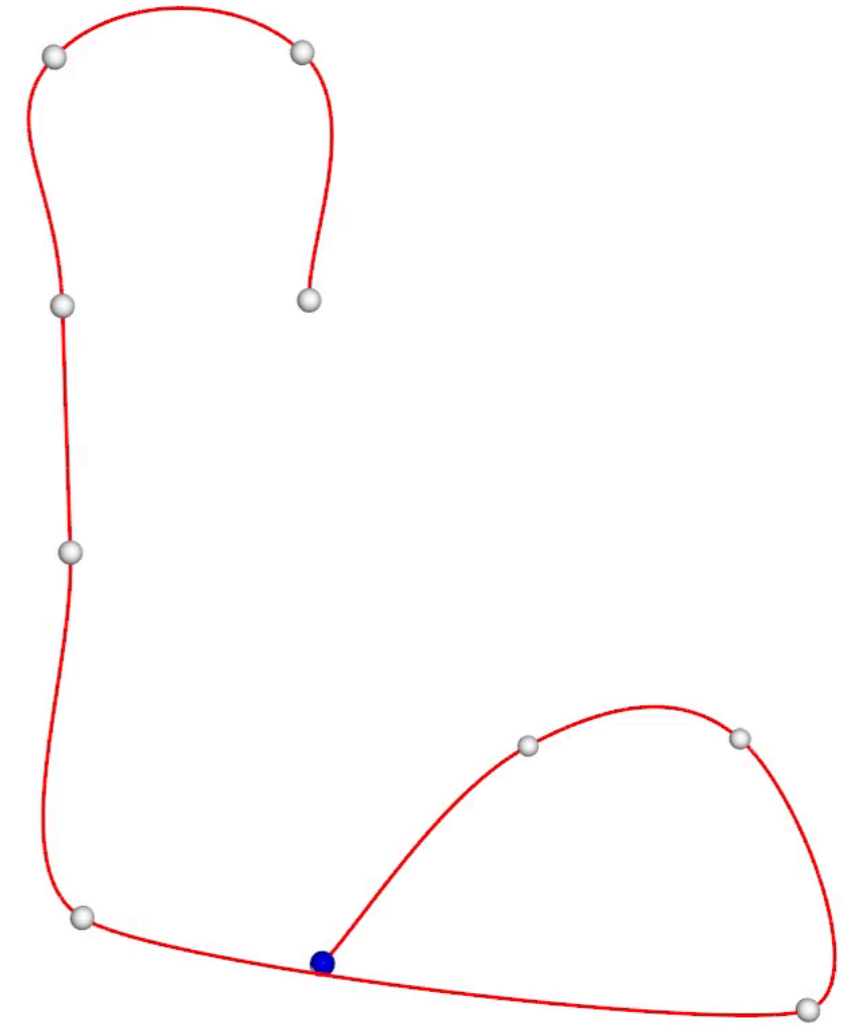
Compute  $p(t)$  as a cubic spline interpolation

- Given keyframes  $(p_i, t_i)_{i \in [0, N-1]}$
- Given time  $t \in [t_1, t_{N-2}]$

1. Find  $i$  such that  $t \in [t_i, t_{i+1}]$

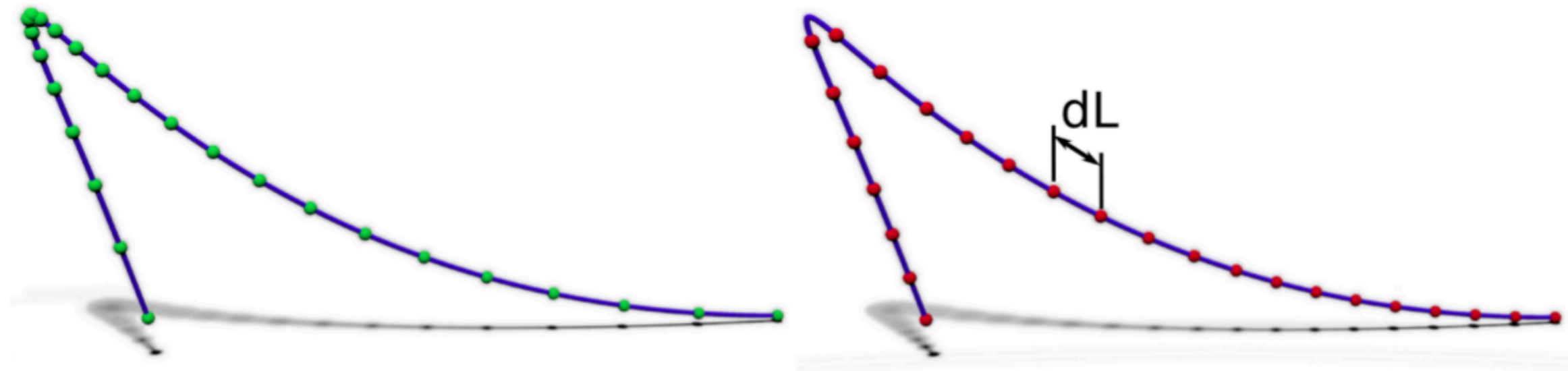
2. Compute  $d_i = \mu \frac{p_{i+1} - p_{i-1}}{t_{i+1} - t_{i-1}}$ , and  $d_{i+1} = \mu \frac{p_{i+2} - p_i}{t_{i+2} - t_i}$

3. Compute  $p(t) = (2s^3 - 3s^2 + 1)p_i + (s^3 - 2s^2 + s)d_i + (-2s^3 + 3s^2)p_{i+1} + (s^3 - s^2)d_{i+1}$   
with  $s = \frac{t - t_i}{t_{i+1} - t_i}$ .



# Limitation of cubic curve interpolation

- Only  $C^1$ , but not  $C^2$  at junctions : Curvature / acceleration discontinuity
  - Force  $C^2$  continuity for cubic polynomial (global linear system to solve - loose local structure)
  - Consider higher degree polynomial
- Non constant speed along each polynomial
  - Reparametrization with curvilinear length



# Curve editing

- Animation software (Maya, 3DSMax, Blender, etc) always come with a **curve editor**.
- Artists can manually adjust their position, time, and **derivatives** on curve editor.
  - One curve for each scalar parameter
    - position  $(x, y, z)$
    - scaling  $(s_x, s_y, s_z)$
    - rotation/ quaternion  $(q_x, q_y, q_z, q_w)$
- Can also use a wrapper function  $w$  to change time  $p(t) = f(w(t))$

