Fundamental models

1- Particles

2- Rigid bodies

3- Continuum models

Rigid body description

- Solid defined within a domain $\Omega \subset \mathbb{R}^3$
- With a density of mass $\rho(p_i)$ at each point $p_i \in \Omega$

- Total mass of the solid m

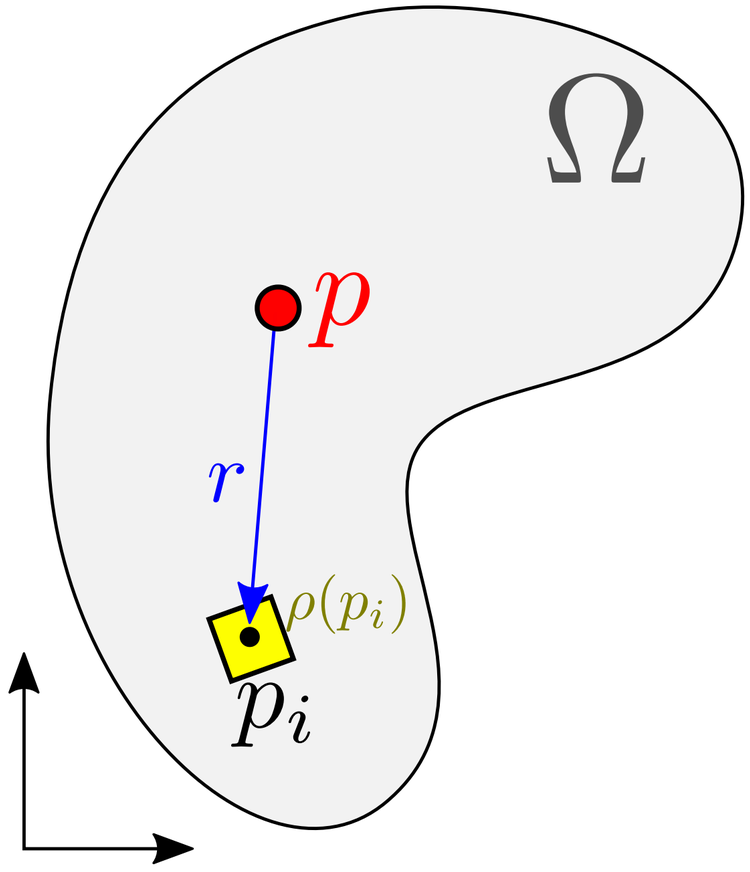
$$m = \int_{p_i \in \Omega} \rho(p_i)\, \mathrm{d}\Omega$$

- Position of the center of mass (com) p

$$p = \frac{1}{m} \int_{p_i \in \Omega} \rho(p_i)\, p_i \, \mathrm{d}\Omega$$

- Relative position of a position p_i with respect to com

$$r = p_i - p$$



Position and speed of a point on the rigid body

The center of mass has, at time t,

- a position $p(t)$
- a velocity $p'(t) = v(t)$

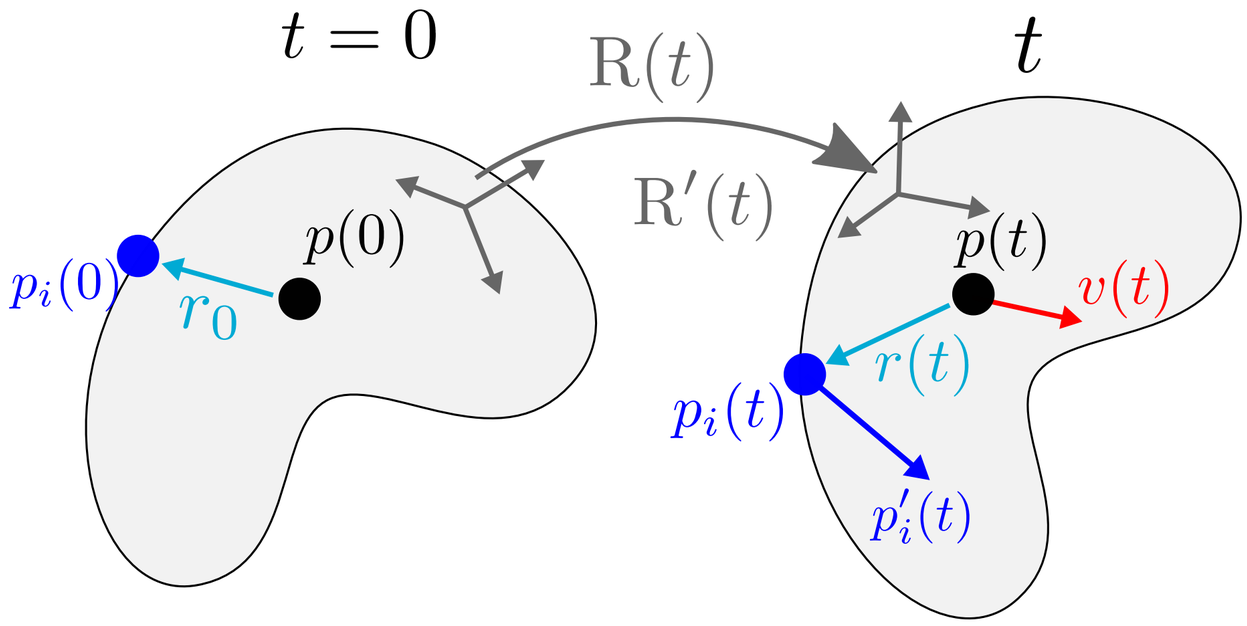
The body has an orientation $\mathrm{R}(t)$

A point of the rigid body has

- a position $\boxed{p_i(t) = p(t) + R(t)\, r_0}$

  with $r_0 = p_i(0) - p(0)$

- a speed $\boxed{p_i'(t) = v(t) + R'(t)\, r_0}$



Angular speed

Speed of p_i : $p_i'(t) = v(t) + \mathrm{R}'(t)\, r_0$

Introduce angular speed $\omega \in \mathbb{R}^3$ such that

$$p_i'(t) = v(t) + \omega(t) \times r(t)$$

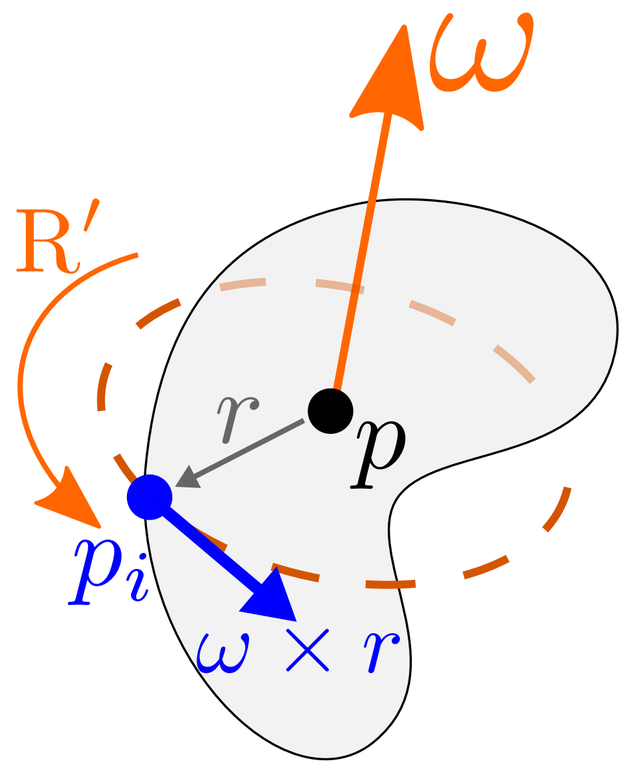
$\omega \simeq$ *vector expressing the instantaneous rotation of* $r(t)$

By identification $\mathrm{R}'(t)\, r_0 = \omega(t) \times r(t)$
$\Rightarrow \mathrm{R}'(t)\, r_0 = \omega(t) \times (\mathrm{R}(t)\, r_0)$

Matrix expression of $\omega = (\omega_x, \omega_y, \omega_z)$

$$\hat{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

$$\Rightarrow \mathrm{R}'(t) = \hat{\omega}(t)\, \mathrm{R}(t)$$



Rigid body kinematics

Similarily to particles

- Position of the com: $p(t)$
- Velocity of the com: $v(t) = p'(t)$

Linear Momentum: $\boxed{P(t) = m\, v(t)}$ $\left(= \int_\Omega \rho\, v_i(t)\, \mathrm{d}\Omega \right)$

Specific to rigid body

- Orientation of the body: $\mathrm{R}(t) \in \mathbb{R}^{3\times 3}$
- Angular velocity of the body: $\omega(t)$ such that $\hat{\omega} = \mathrm{R}'(t)\, \mathrm{R}^T(t)$

Angular Momentum: $\boxed{L(t) = \mathrm{I}(t)\, \omega(t)}$

with $\mathrm{I}(t)$: Inertia tensor

$$\mathrm{I}(t) = \mathrm{R}(t)\, \mathrm{I}_0\, \mathrm{R}^T(t)$$

$$\mathrm{I}_0 = \int_{r\in\Omega} \rho(r)\, (r^T r\, \mathrm{I}_d - r\, r^T)\, \mathrm{d}\Omega$$

Mass: resistance to change of speed (of the com)

Inertia: resistance to change of angular speed

Inertia tensor

Defining inertia tensor formulation from angular momentum definition

Angular momentum expressed with respect to an arbitrary point p_0: $r(p_i) = p_i - p_0$

$$L = \int_\Omega r \times (\rho\, r')\, \mathrm{d}\Omega = \int_\Omega \rho\, r \times (p' + \omega \times r)\, \mathrm{d}\Omega \quad \textit{(first part sum to 0)}$$

$$\Rightarrow L = \int_\Omega \rho\, r \times \omega \times r\, \mathrm{d}\Omega = \int_\Omega \rho\, r \times (-r \times \omega)\, \mathrm{d}\Omega$$

$$\Rightarrow L = \underbrace{\left(\int_\Omega \rho\, \hat{r}\, \hat{r}^T \mathrm{d}\Omega\right)}_{I} \omega \;=\; I\,\omega \quad \text{with } \hat{r} = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix}$$

Changing the reference frame

$$\Rightarrow L = \left(\int_\Omega \rho\, (\mathrm{R}\hat{r}_0)\, (\mathrm{R}\hat{r}_0)^T \mathrm{d}\Omega\right)\, \omega \;=\; \mathrm{R}\underbrace{\left(\int_\Omega \rho(r)\, \hat{r}_0\, \hat{r}_0^T \mathrm{d}\Omega\right)}_{I_0}\mathrm{R}^T\, \omega = \mathrm{R}\, I_0\, \mathrm{R}^T\, \omega$$

Inertia tensor properties

$$I = \int_{r \in \Omega} \rho(r) \begin{pmatrix} r_y^2 + r_z^2 & -r_x r_y & -r_x r_z \\ -r_x r_y & r_x^2 + r_z^2 & -r_y r_z \\ -r_x r_z & -r_y r_z & r_x^2 + r_y^2 \end{pmatrix} \mathrm{d}\Omega \quad = \quad \int_{r \in \Omega} \rho(r) \left(r^T r \, \mathrm{Id} - r \, r^T \right) \mathrm{d}\Omega$$

- I is usually expressed at the center of mass p

- I depends on the body orientation. Given a rotation R: $I = \mathrm{R} \, I_0 \, \mathrm{R}^T$
  $\Rightarrow$ compute once I_0 in a rest position, then update it using R

- There exist a frame in which I is diagonal (principle axes of inertia).
  Corresponds to eigenvectors of matrix I.

Dynamics: Forces and torques on a solid

- Given a local force $f(p_i)$ acting on a position $p_i \in \Omega$
- Contribute to 2 global components applied on the body:

  - Total **net force** F applied on the shape

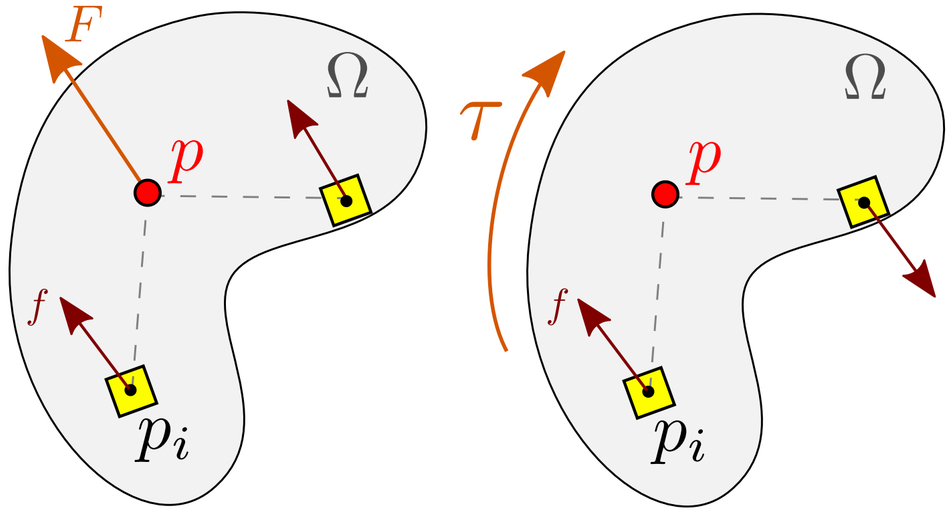
    Induce change of linear momentum (a displacement of COM)

$$F = \int_{p_i \in \Omega} f(p_i)\, \mathrm{d}\Omega$$

  - **Torque** τ applied on the body

    Induce change of angular momentum (spin of the solid)

$$\tau = \int_{p_i \in \Omega} (p_i - p) \times f(p_i)\, \mathrm{d}\Omega$$



Dynamics

Similarly to particles

Force F is related to the change of linear momentum $F(t) = P'(t) = (m\,v)'(t)$

Specific to rigid bodies

Torque τ is related to the change of angular momentum $\tau(t) = L'(t) = (I\,\omega)'(t)$

Equation of Motion

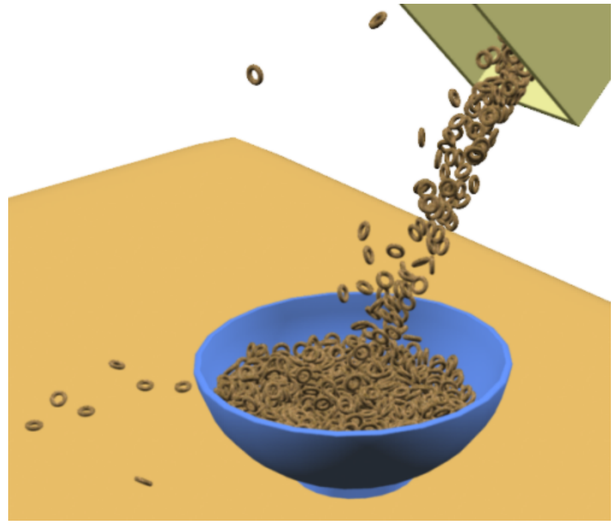
Fundamental principle of dynamics for rigid body

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p \\ P \\ \mathrm{R} \\ L \end{pmatrix} = \begin{pmatrix} v \\ F \\ \hat{\omega}\ \mathrm{R} \\ \tau \end{pmatrix}$$

Rigid solid dynamics in practice

1. Initial condition

- $p(t_0), v(t_0), \mathrm{R}(t_0), \omega(t_0)$ given as initial condition
- Precompute $I_0 = I(t_0)$
- Compute $L(t_0) = I_0\ \omega(t_0)$



2. Temporal Evolution

Iterate over time t_k

- Compute total force $F(t_k)$ and torque $\tau(t_k)$
- Compute $I(t_k) = \mathrm{R}(t_k)\, I_0\, \mathrm{R}^T(t_k)$
- Compute $\omega(t_k) = I(t_k)^{-1}\, L(t_k)$

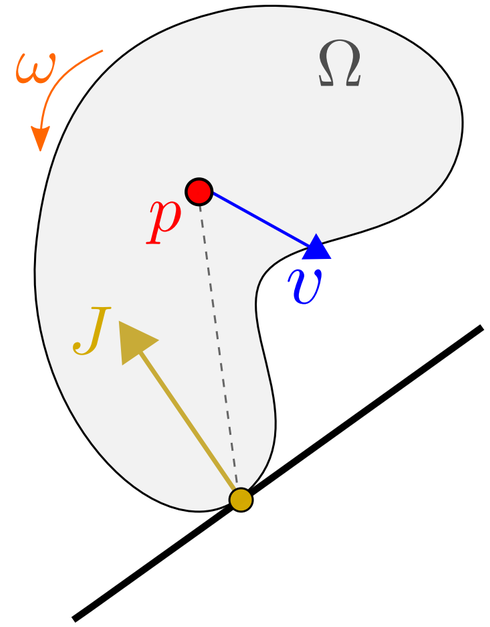
- Numerical integration updating state vector
  $\rightarrow (p(t_{k+1}), P(t_{k+1}), \mathrm{R}(t_{k+1}), L(t_{k+1}))$

- *Handle collision*

$$\begin{pmatrix} p'(t) \\ P'(t) \\ \mathrm{R}'(t) \\ L'(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t) \\ \hat{\omega}(t)\ \mathrm{R}(t) \\ \tau(t) \end{pmatrix}$$

$$I(t) = \mathrm{R}(t)\, I_0\, \mathrm{R}^T(t)$$
$$L(t) = I(t)\, \omega(t)$$
$$P(t) = m\, v(t)$$

Collisions in rigid bodies



Collisions at position p change both linear and angular velocity

Use of impulse (sudden change of velocity) J.

$$J = \int_{t_0}^{t_0+\Delta t} F(t)\mathrm{d}t$$

Impulse split into

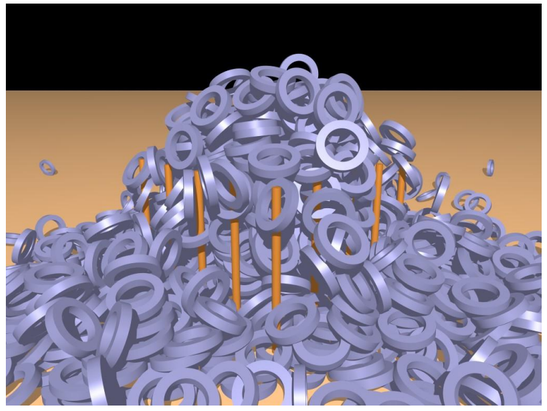
- *Force impulse* $\Delta P = m\,\Delta v = J$
- *Torque impulse* $\Delta L = I\,\Delta\omega = (p - p_{com}) \times J$

Elastic collision between two solids at position p_i:

$J = j\,n$, n: normal of the separating planes

$j = (v_1(p_i) - v_2(p_i)) \cdot n / K$

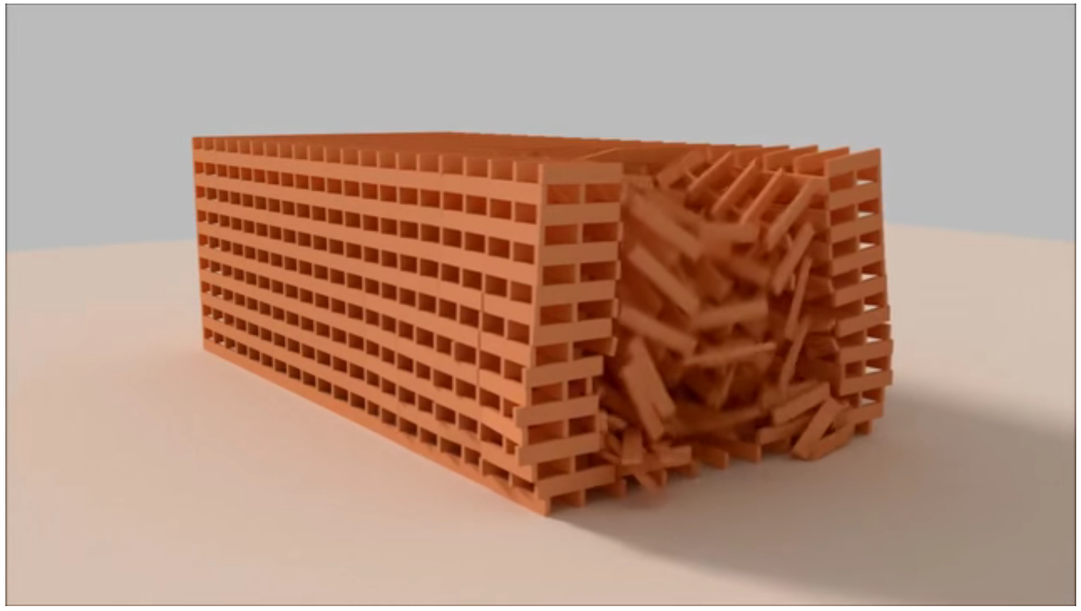
$K = 1/m_1 + 1/m_2 + n \cdot \left(I_1^{-1}(r_1 \times n) \times r_1 + I_2^{-1}(r_2 \times n) \times r_2\right)$



More details [D. Barff. Physically Based Modeling. SIGGRAPH Course Notes 1999]

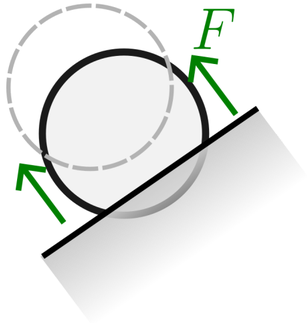
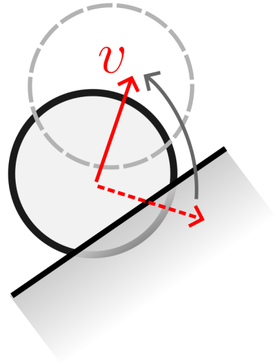
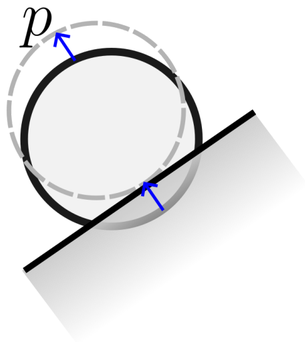
Rigid bodies usage

- Standard usage for rigid bodies motions
  - Limited to non-deformable shapes
- Common in VFX (explosions), and *simulation games* (cars, airplanes, etc).
- Standard library: Bullet physics (ex. used in Blender).



Position Based Dynamics (PBD)

Constraint handling

Force based	Impulse based	Position based
		
$v_i^{k+1} = v_i^k + F_i/m_i\,\Delta t$	$v_i^{k+1} = v_i^k + J/m$	$p_i^{k+1} \rightarrow \mathcal{F}(p^k)$
(+) Physical interpretation (+) Arbitrary constraint (-) Constraint stiffness: Divergence	(+) Stability (-) Collision only (-) Doesn't correct existing collisions	(+) Stability (+) Arbitrary constraint (-) Physical interpretation

PBD: General Idea

Handle positional constraints by moving vertex positions directly

Velocity computed as change of position

(+) Very simple, unconditionally stable

Formalized in 2005

[Matthias Muler, Bruno Heidelberger, Marcus Hennix, John Racliff. Position based Dynamics. PRIPHYS 2005.]

PBD simulation step

1) Integrate *standard forces* to predict position

$p_{\text{prev}} = p$

$v = v + \Delta t \, F/m$

$p = p + \Delta t \, v$

2) Project p on constraints

3) Update velocity

$v = (p - p_{\text{prev}})/\Delta t$

PBD: Generic Non Linear Constraints

Handling arbitrary non constraint $C(p) > 0$ over $p = (p_1, \cdots, p_N)$

Solve iteratively $C(p + \mathrm{d}p) \simeq C(p) + \nabla C \cdot \mathrm{d}p > 0$ using Lagrange multiplier

$\mathrm{d}p = \lambda M^{-1} (\nabla C)^T$, $M = \mathrm{diag}(m_1, \cdots, m_N)$:masses

$\Delta p_i = -\lambda / m_i \nabla_{p_i} C(p)^T$

$$\Rightarrow \lambda = \frac{C(p)}{\sum_i 1/m_i \left\| \nabla_{p_i} C(p) \right\|^2}$$

Note: Converges to infinite stiffness

Proposition of eXtended PBD (XPBD)

$$\Rightarrow \lambda = \frac{C(p)}{\sum_i 1/m_i \left\| \nabla_{p_i} C(p) \right\|^2 + \frac{\alpha}{(\Delta t)^2}}$$

$\alpha = 1/K$: compliance (inverse stiffness)

[Miles Macklin, Matthias Muller, Nuttapong Chentanez. XPBD: Position-Based Simulation of Compliant Constrained Dynamics. MIG 2016]

[Jan Bender, Matthias Muller, Miles Macklin. A survey on Position Based Dynamics. EUROGRAPHICS STAR 2017]

Note: Formulation may vary a bit depending on the paper.

Rigid bodies dynamic: Shape Matching

General Idea:

1) Consider a reference shape.

2) At time t, deform vertices individually

$t \rightarrow t + \Delta t$

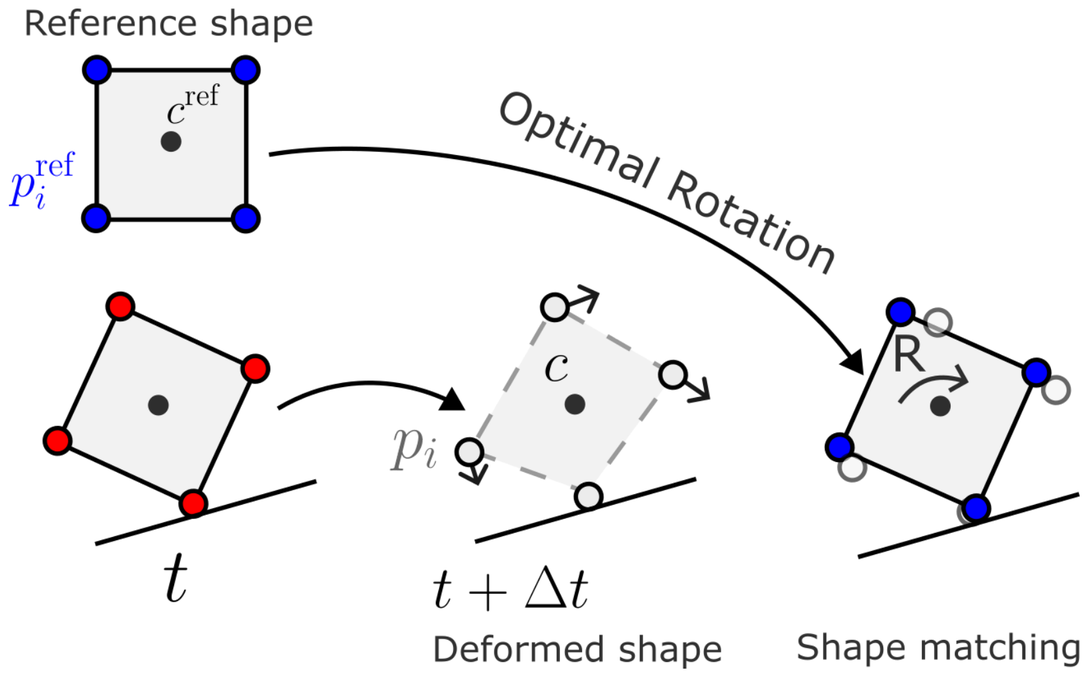
Solve for the collision constraints on each vertex

3) Shape Matching:

Compute optimal rotation R

between reference shape and current one.

Apply R on the reference shape (+ translate it).



(+) Garantee to preserve reference shape appearance

(-) May not preserve individual constraints

Shape Matching Formulation

Reference Shape:

- Vertex position p_i^{ref}
- Center of mass c^{ref}
- Local vertex vector $r_i^{\text{ref}} = p_i^{\text{ref}} - c^{\text{ref}}$

Deformed shape at $t + \Delta t$

- Vertex position p_i
- Center of mass c
- Local vertex vector $r_i = p_i - c$

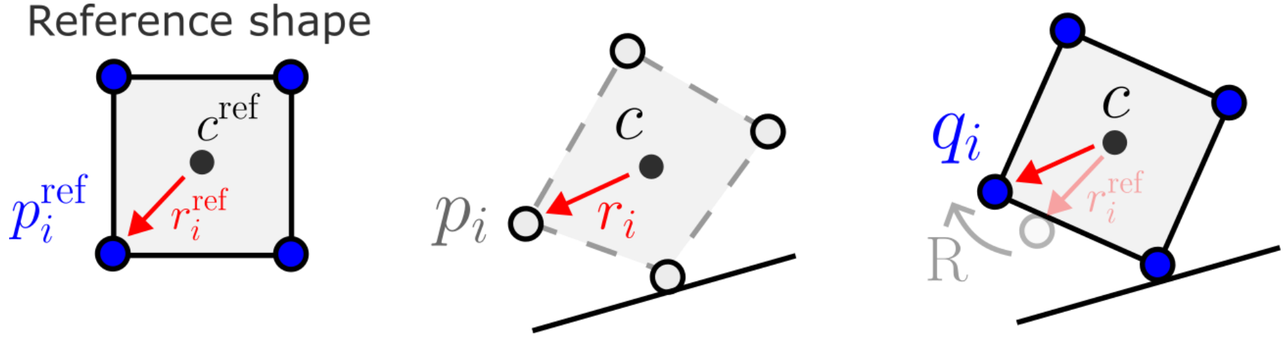
Averaged transformation $\mathrm{T} = \sum_i r_i \left(r_i^{\text{ref}}\right)^T$

- T minimizes $\sum_i \|\mathrm{T}\, r_i^{\text{ref}} - r_i\|^2$

Extract optimal rotation R

- via polar decomposition from T

New position $q_i = \mathrm{R}\, r_i^{\text{ref}} + c$



Reminder Polar Decomposition: $\mathrm{R} = WV^T$, for $svd(T) = W\,\Sigma\,V^T$

Polar Decomposition

Goal: Decompose a 3×3 matrix M into

- rotation part R
- scaling/shearing D

ex. Usefull for interpolation of transformations.

- Interpolate rotation with quaternion (ex. SLERP/LERP)
- Scaling/shearing using componentwise interpolation (ex. linear).

$\Rightarrow$ Polar decomposition: $\mathrm{M} = \mathrm{R}\,\mathrm{D}$

- R: Rotation matrix
- D: Positive semi-definite matrix

[K. Shoemake and T. Duff, Matrix Animation and Polar Decomposition. Graphics Interface 92.]

- Polar decomposition is obtained from SVD

$$\mathrm{SVD}(\mathrm{M}) = W \Sigma V^T \text{ with } \boxed{\mathrm{R} = \mathrm{W}\mathrm{V}^T}, \quad \mathrm{D} = \mathrm{V}\Sigma\mathrm{V}^T$$

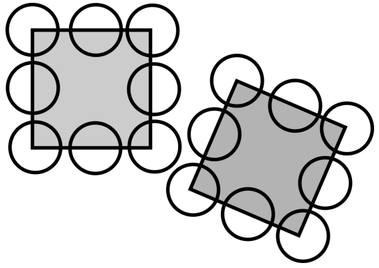
- Or numerically, R can be computed using the following iterative scheme

$$\boxed{R_0 = \mathrm{M}\,,\ R_{i+1} = 0.5\,(R_i + (R_i^{-1})^T)}$$

Shape Matching Usage

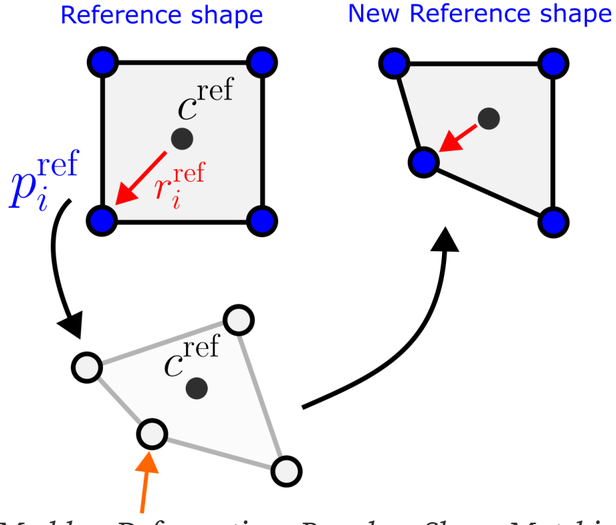
Rigid bodies

Collision b/w bodies using colliders around vertices



Plastic deformation

Update the reference shape for large deformation



[Muller et al. Meshless Deformations Based on Shape Matching. SIGGRAPH 2005]

[Macklin et al. Unified Particle Physics for Real-Time Applications. SIGGRAPH 2014]